Duality and entanglement in lattice gauge theories

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Based on:

AB and M. Panero *JHEP* 06 (2023) 030
AB and M. Panero *JHEP* 06 (2024) 041

QuantFunc, Valencia, 2nd September 2024

Motivations

The study of entanglement in many-body systems has applications in different areas of physics:

- Quantum phases of matter and quantum phase transitions [Vidal et al.; quant-ph/0211074].
  - **Effective number** of degrees of freedom [Casini, Huerta; hep-th/0405111].
- High-energy physics.
  - **Confinement** [Klebanov et al.; 0709.2140].
- AdS/CFT and quantum gravity [Ryu, Takayanagi; hep-th/0603001].
- Quantum computing and quantum simulations [Abanin, Demler; 1204.2819] [Daley et al.; 1205.1521].

Dirac Medallists 2024

Horacio Casini, Marina Huerta, Shinsei Ryu, Tadashi Takayanagi

For "*pioneering contributions to the understanding of quantum entropy in gravity and quantum field theory*".

Motivations

In order to study entanglement measures one has to face different challenges.

- Typical numerical techniques **struggle to compute** such highly non-local quantities.

↓

- New, efficient numerical techniques (1st part of the talk).

- For gauge theories, the **definition** itself of entanglement is **ambiguous** due to the Gauß law.

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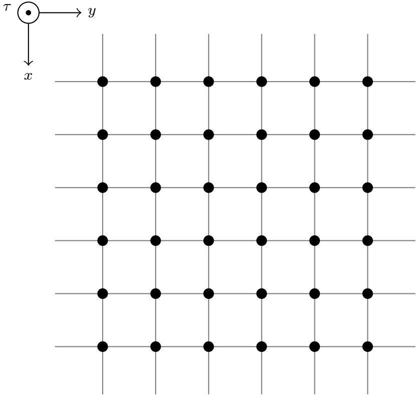
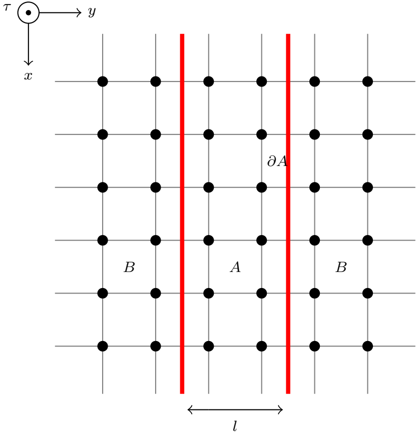
- Better understanding on how to treat entanglement in presence of local constraints (2nd part of the talk).

Entanglement in QFT



Entanglement in QFT

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$



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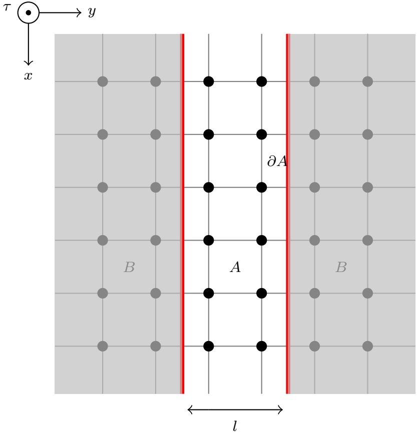
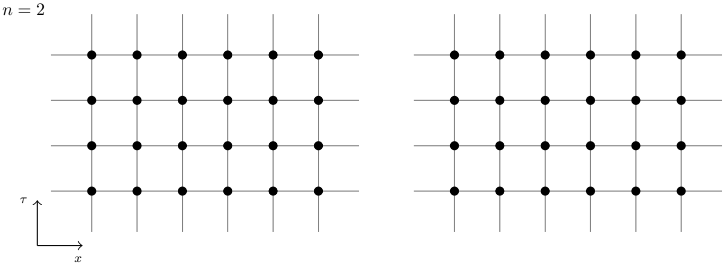


$$S = -\operatorname{Tr}(\rho_A \log \rho_A) \qquad S_n = \frac{1}{1-n} \log \operatorname{Tr} \rho_A^n \qquad C_n = \frac{l^{D-1}}{|\partial A|} \frac{\partial S_n}{\partial l}$$

- A common way to calculate $\mathrm{Tr}\,\rho_A^n$ is to exploit the replica trick [Calabrese, Cardy; hep-th/0405152] (lattice discretization: [Buividovich, Polikarpov; 0802.4247]).



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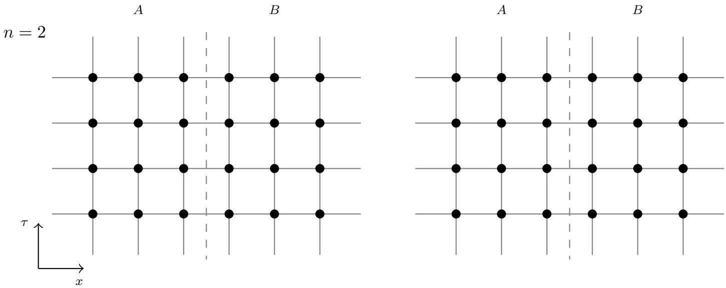

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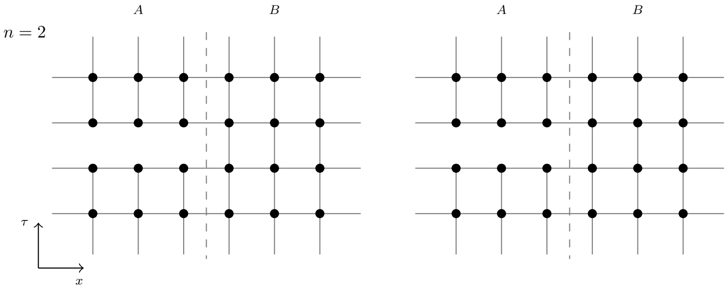

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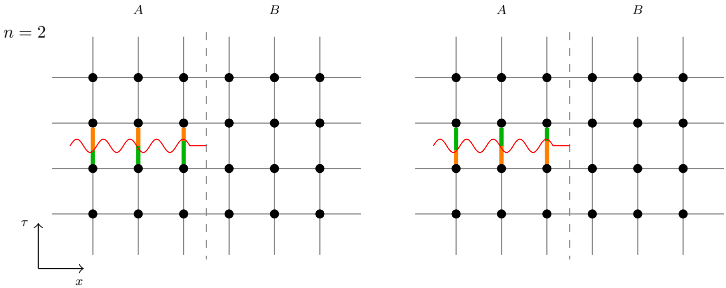

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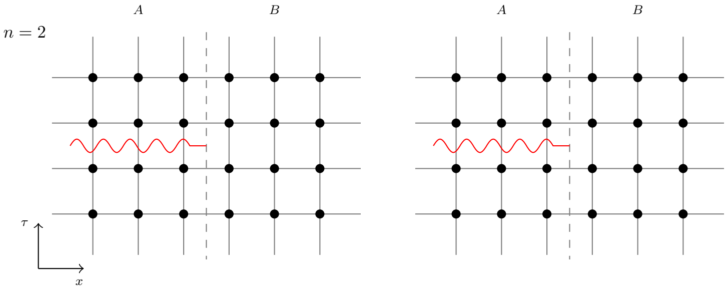

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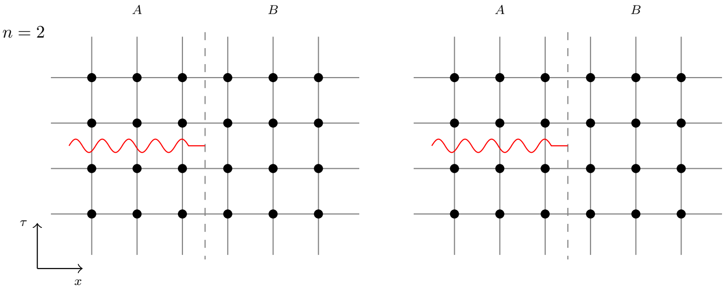
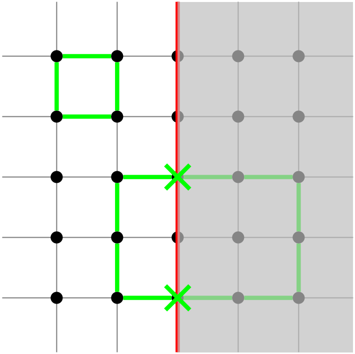


$$S_n = \frac{1}{1-n}\log\frac{Z_n}{Z^n} \qquad C_n = \frac{l^{D-1}}{|\partial A|}\frac{\partial S_n}{\partial l} \sim \frac{1}{a}\log\frac{Z_n(l+a)}{Z_n(l)}$$

- Monte Carlo simulations can compute $Z_n(l+a)/Z_n(l)$ in arbitrary dimension ...
- ... but **typical methods** are **not efficient** in doing so!

$$\mathcal{H}_{\text{phys.}} \neq \mathcal{H}_{\text{phys.},A} \otimes \mathcal{H}_{\text{phys.},B}$$



The problems to face

1st part of the talk

$$C_n \to Z_n(l+a)/Z_n(l)$$

- Common Monte Carlo methods to compute rations of partition functions can be **biased** or suffer from **bad signal-to-noise ratio**.

↓

- Non-equilibrium Monte Carlo simulations for entanglement measures.

2nd part of the talk

$$\mathcal{H}_{\text{phys.}} \neq \mathcal{H}_{\text{phys.},A} \otimes \mathcal{H}_{\text{phys.},B}$$

- The non-factorizability of the Hilbert space leads to an **ill-defined** replica geometry.

↓

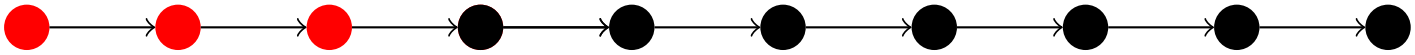
- Duality transformations, mapping Abelian gauge theories to spin models.

Non-equilibrium Monte Carlo for entanglement measures

Markov Chain Monte Carlo simulations

- Monte Carlo simulations allow for calculations of expectation values of observables sampling from Boltzmann distributions

$$\langle O \rangle = \sum_i O_i \frac{e^{-S}}{Z}$$



Thermalization — Equilibrium configurations $\rightarrow$ Measurements

Problems

- Critical slowing down
- Sign problem (no real-time dynamics)

Advantages

- Efficient calculations of ground state, local observables
- **Scaling**

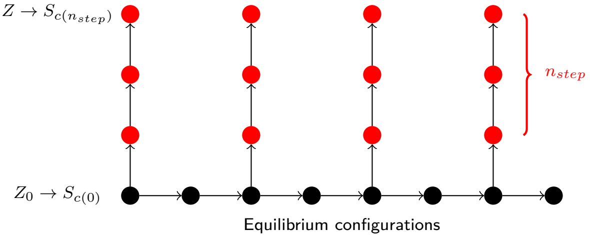
- However, equilibrium Monte Carlo techniques fail to efficiently compute non-local quantities

$$\frac{Z}{Z_0} = \langle e^{-\Delta S} \rangle \quad \longrightarrow$$ **Exponential** scaling in the volume!

Non-equilibrium Monte Carlo: the Jarzynski's equality

[Jarzynski; cond-mat/9610209]

$$q_0 \sim e^{-S_0} = e^{-S_{c(0)}} \to e^{-S_{c(1)}} \ldots \to e^{-S_{c(n_{step})}} = e^{-S} \sim p$$

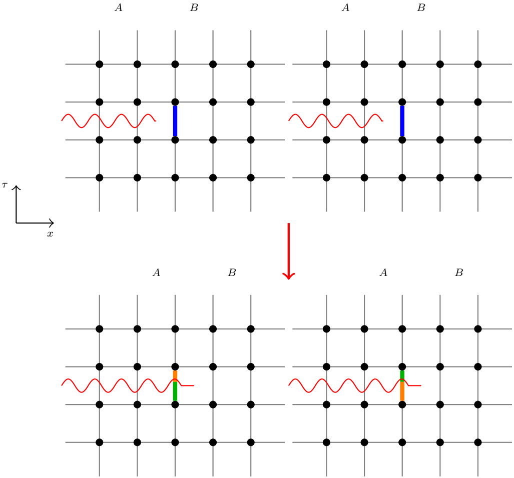


$W = \sum_{n=0}^{n_{step}} \left\{ S_{c(n+1)} - S_{c(n)} \right\} \longrightarrow$ Jarzynki's equality: $\langle \exp(-W) \rangle = Z/Z_0$

Polynomial scaling in the number of d.o.f.: [Caselle et al.; 1801.03110], [AB, Cellini, Nada; WIP].

Non-equilibrium Monte Carlo for entanglement entropy

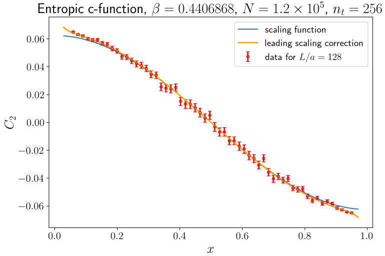
- Protocol introduced in [Alba; 1609.02157] and generalized in [AB, Panero; 2304.03311].
- Applied to spin models (e.g. $S = -\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j$), generalizable to gauge theories.

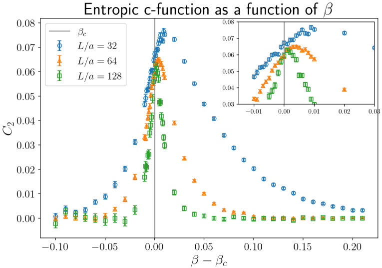


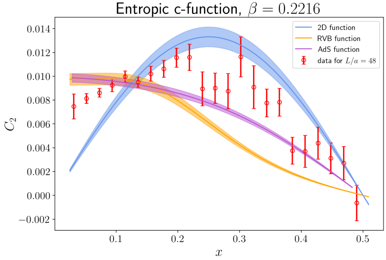
$$\begin{cases} \beta = \beta \\ \beta = 0 \\ \beta = 0 \end{cases} \longrightarrow \begin{cases} \beta = 0 \\ \beta = \beta \\ \beta = \beta \end{cases}$$

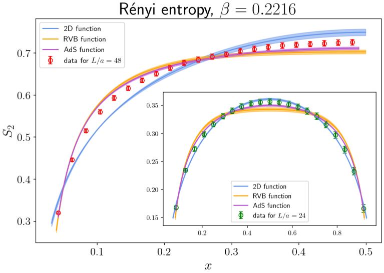
$$Z_2(l) \longrightarrow Z_2(l + a)$$

Some results for the $(1+1)D$ and $(2+1)D$ Ising model









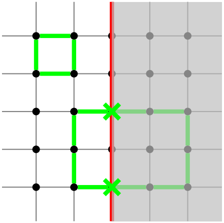
[AB, Panero; 2304.03311]

Gauge theories: duality and entanglement

$$\mathcal{H}_{\text{phys.}} \neq \mathcal{H}_{\text{phys.},A} \otimes \mathcal{H}_{\text{phys.},B}$$

$$\downarrow$$

The replica space is ill defined!

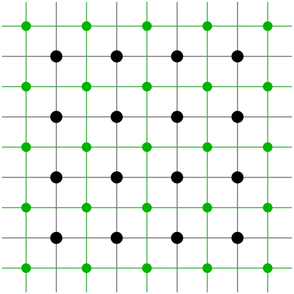


The problem has been addressed in a variety of ways and theories:

- Local extension of the Hilbert space [Buividovich, Polikarpov; 0806.3376], [Donnelly; 1109.0036],[Ghosh et al.; 1501.02593], [Soni, Trivedi; 1510.07455].
- Algebraic approach [Casini et al.; 1312.1183].
- Momentum-basis factorization [Aoki et al.; 1502.04267].
- New lattice geometries [Chen et al., 1503.01766].
- **Lattice dualities** [Casini, Huerta; 1406.2991], [Radičević; 1605.09396], [Lin, Radičević; 1808.05939], [Moitra et al.; 1811.06986].

Kramers-Wannier duality

- Kramers-Wannier duality [Kramers, Wannier; 1941] relates two different lattice models: $Z \propto Z_{\text{dual}}$.
- In three spacetime dimensions it maps a **gauge theory** to a **spin model**.
- In particular: Ising $3D \leftrightarrow \mathbb{Z}_2$ gauge theory.



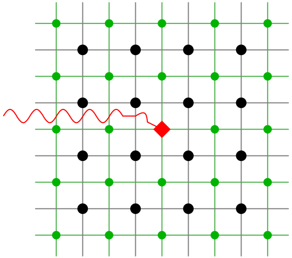
- It was shown that the universal information encoded in the entanglement entropy is **exactly mapped** [Casini, Huerta; 1406.2991], [Moitra, Soni, Trivedi; 1811.06986], in particular

$$C_n^{\text{gauge}}(\beta^\star, l) = C_n^{\text{spin}}(\beta, l)$$

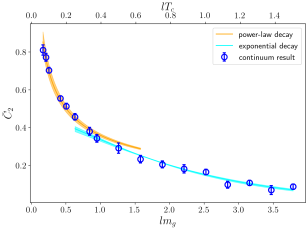
Duality and entanglement

For gauge theories that admit a dual spin model,
it is possible to:

- Explicitly derive the geometry of the replica space for the gauge theory.



- Simulate the spin model to compute the entropic c-function of the gauge theory.



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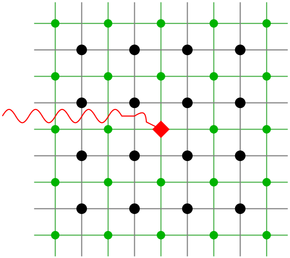

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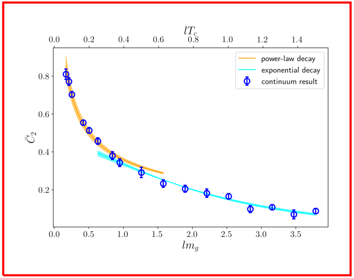
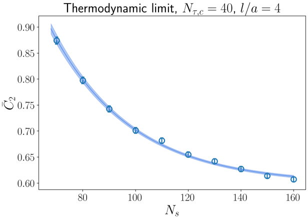


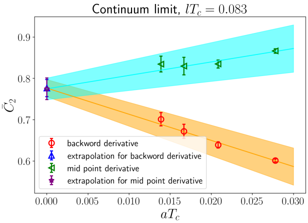
Setup

- In [AB, Panero; 2404.01987] we computed C_2 in the $(2+1)$-**dimensional** $\mathbb{Z}_2$ **gauge theory** exploiting duality.
- **Non-equilibrium MC** introduced in [Alba; 1609.02157], [AB, Panero; 2304.03311].
- We studied the **confined phase** using the scale setting in [Caselle, Hasenbusch; hep-lat/9511015].
- Simulated volumes up to $210^2 \times 1000$ (zero-temperature/ground state results).

Extrapolations

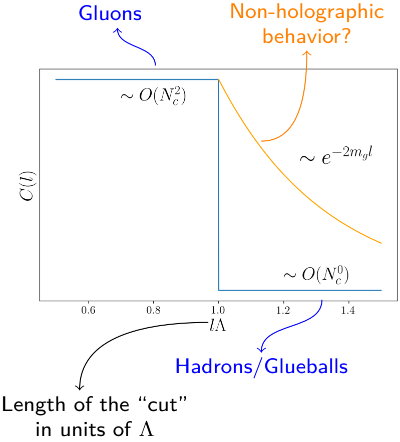
- **First thermodynamic and continuum** extrapolation of a $(2+1)$-dimensional entropic c-function in literature.



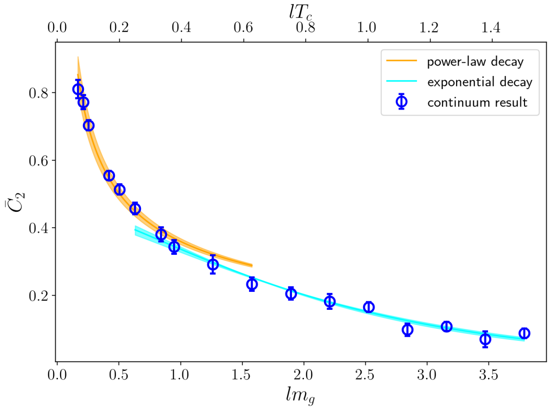


Theoretical predictions

- The entropic c-function detects the **effective number** of (IR) d.o.f. of a theory.
- In confining, holographic theories the entropic c-function has a **sharp transition** [Klebanov et al.; 0709.2140]
  $\longrightarrow$ entanglement as a probe of confinement.
- The scale of the transition is $\Lambda \sim \Lambda_{QCD} \sim T_c \sim m_{\rm g}$.
- For non-holographic theories **numerical simulations** are required.



Our results



$$f(lm_g; B, c) = \frac{B}{(lm_g)^c} \quad B = 0.360(9) \quad c = 0.48(2)$$ [Florio; 2312.05298]

$$f(lm_g; A, \alpha) = A lm_g \int \mathrm{d}t \exp\left(-2\sqrt{1+t^2}\alpha lm_g\right) \quad A = 0.33(3) \quad \alpha = 0.360(19)$$

[Klebanov et. al.; 0709.2140]

[AB, Panero; 2404.01987]

Conclusions and outlook

Non-equilibrium MC

- Competitive tool for **large scale simulations** of entanglement measures in arbitrary dimension.
- So far applied in the Lagrangian formalism $\rightarrow$ it is possible to take the **Hamiltonian limit** [Funcke et al.; 2212.09627] to study Hamiltonian systems.
- Promising extensions using **(Stochastic) Normalizing Flows** [AB, Cellini, Jansen, Kühn, Nada, Nakajima, Nicoli, Panero; WIP].

Entanglement in gauge theories

- **Unambiguous** results for entanglement entropy in Abelian gauge theories.
- Directly generalizable to other theories, e.g. $U(1)$ gauge theory.
- The **replica geometry** can be derived using the duality: extension to the **non-Abelian** case?
- **Long term goal**: $SU(N)$ gauge theories $\rightarrow$ better understanding of **confinement** via entanglement.

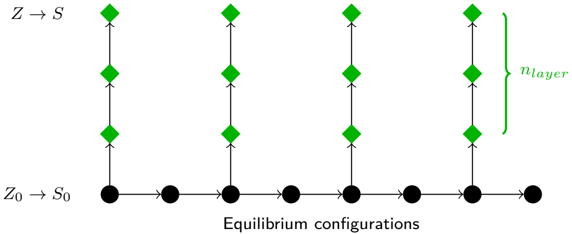
Backup slides

Normalizing flows

- NF are parametric, invertible and differentiable maps between probability distributions.
- Usually consist in a sequence of **coupling layers** with a tractable Jacobian of the transformation [Albergo et al.; 1904.12072], [Kanwar et al.; 2003.06413].

$$g_\theta : q_0 \to q_\theta \simeq p \qquad g_\theta = g_N \circ \ldots \circ g_2 \circ g_1 \qquad q_\theta(\phi) = q_0(g^{-1}(\phi))|\det J_g|^{-1}$$

- Trained minimizing the Kullback-Leibler divergence $D_{KL}(q_\theta||p) = \int \mathrm{d}\phi \, q_\theta(\phi) \log\left(\frac{q_\theta(\phi)}{p(\phi)}\right)$

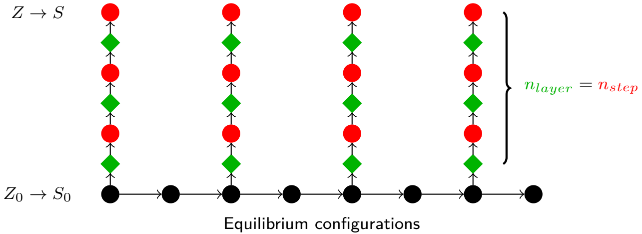


$$W = S - S_0 - \sum_{n=1}^{n_{layer}} \log|\det J_{g_n}| \longrightarrow \boxed{\langle \exp(-W) \rangle = Z/Z_0}$$

[Nicoli et al.; 2007.07115]

- SNF combine stochastic and deterministic updates [Wu et al.; 2002.06707] [Caselle et al.; 2201.08862].
- Trained by minimizing the dissipated work $W - \Delta F \geq D_{KL}(q_0||p)$.

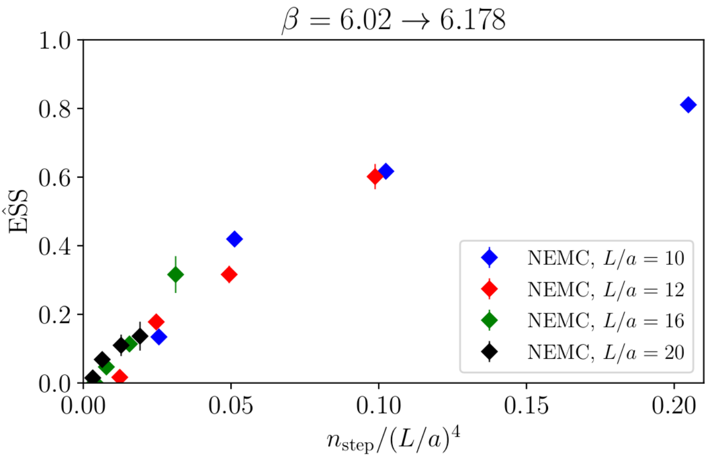
$$\xrightarrow{q_0} \phi_0 \xrightarrow{g_1} g_1(\phi_0) \xrightarrow{e^{-S_{c(1)}}} \phi_1 \xrightarrow{g_2} g_2(\phi_1) \xrightarrow{e^{-S_{c(2)}}} \phi_2 \xrightarrow{g_3} \dots \xrightarrow{e^{-S_{c(n_{step})}}} \phi_{n_{step}}$$



$$W = \sum_{n=0}^{n_{step}} \{S_{c(n+1)} - S_{c(n)}\} - \sum_{n=1}^{n_{layer}} \log|\det J_{g_n}| \longrightarrow \boxed{\langle \exp(-W) \rangle = Z/Z_0}$$

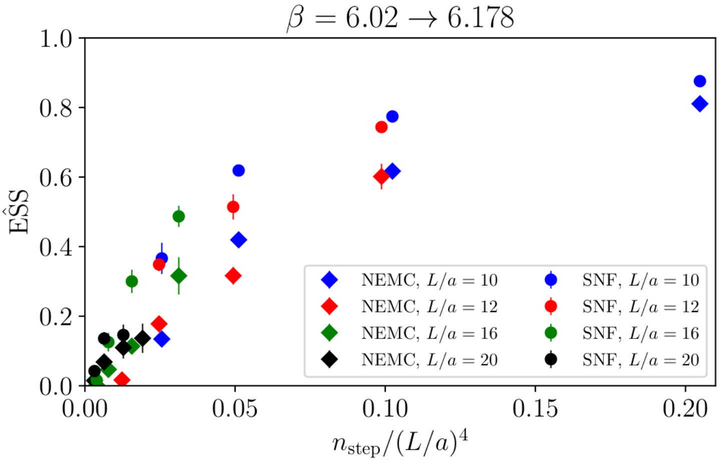
Inherited **scaling with #d.o.f.**: [AB, Cellini, Nada; WIP].

Non-equilibrium MC for $SU(3)$ in $(3+1)D$: scaling



[AB, Cellini, Nada; WIP]

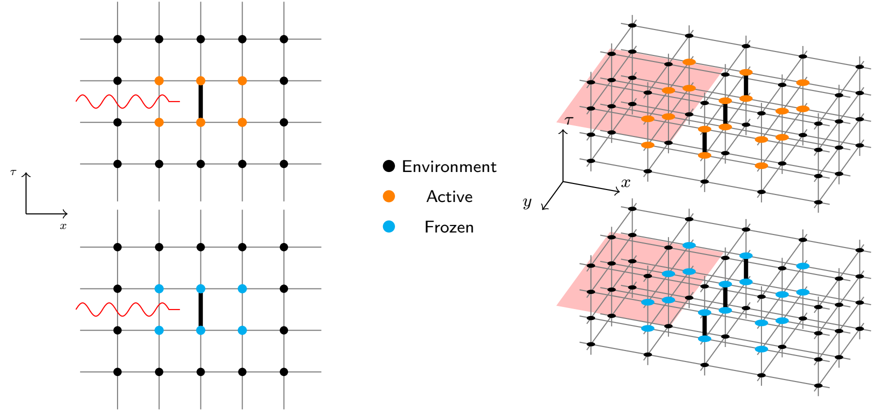
SNF for $SU(3)$ in $(3+1)D$: scaling



[AB, Cellini, Nada; WIP]

Defect coupling layers

Target theory: $S = \sum_{i,\mu} -2\kappa\, \phi_i \phi_{i+\hat{\mu}} + (1-2\lambda)\phi_i^2 + \lambda \phi_i^4$



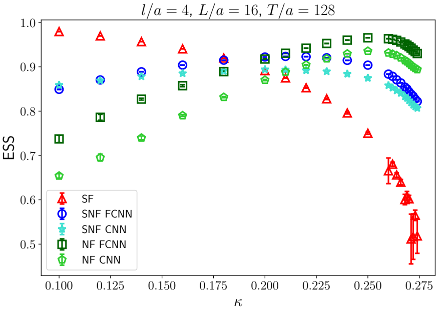
Neural network outputs

$\phi'_{\text{active}} = \exp(-|s(\phi_{\text{frozen}})|)\phi_{\text{active}} + t(\phi_{\text{frozen}})$ $\mathbb{Z}_2$ equivariant [Del Debbio et al.; 2105.12481].

Related works: flows for lattice defects [Abbott et al.; 2404.11674], autoregressive neural networks for entanglement entropy [Białas et al.; 2406.06193].

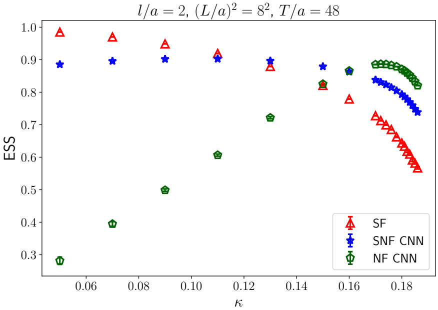
Transfer in the coupling

$(1+1)D$



- NE-MCMC: $n_{step} = 2$
  SNF: $n_{step} = n_{layer} = 2$
  NF: $n_{layer} = 4$

$(2+1)D$

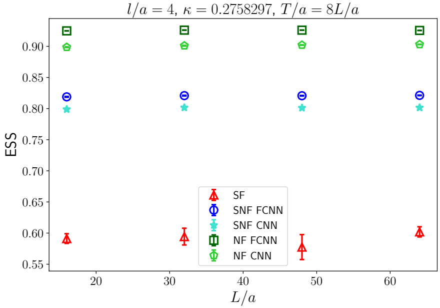


- NE-MCMC: $n_{step} = 5$
  SNF: $n_{step} = n_{layer} = 5$
  NF: $n_{layer} = 5$

[AB, Cellini, Jansen, Kühn, Nada, Nakajima, Nicoli, Panero; WIP]

Transfer in the volume

$(1+1)D$



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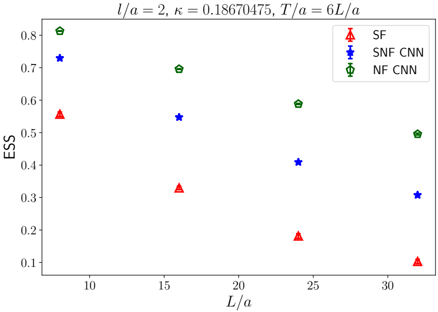
$(2+1)D$



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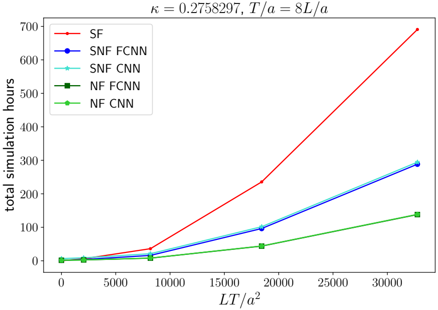
[AB, Cellini, Jansen, Kühn, Nada, Nakajima, Nicoli, Panero; WIP]

Total cost of the simulations

$(1+1)D$



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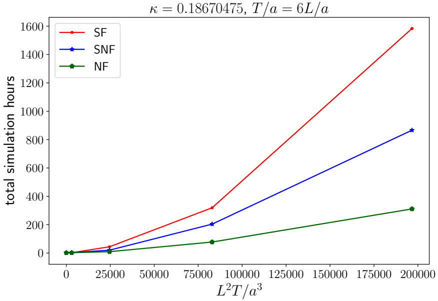
  NF: $n_{layer} = 4$

$(2+1)D$



- NE-MCMC: $n_{step} = 5$
  SNF: $n_{step} = n_{layer} = 5$
  NF: $n_{layer} = 5$

[AB, Cellini, Jansen, Kühn, Nada, Nakajima, Nicoli, Panero; WIP]

Duality and entanglement

Spin model

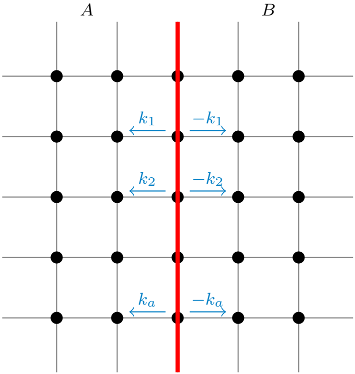
(e.g., $D = 3$ Ising model)

Dual model

(e.g., $D = 3$ $\mathbb{Z}_2$ gauge theory)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Lagrangian} & Z_n \arrow[r, "\text{[AB, Panero; 2404.01987]}"] & Z_n^\star \\
\text{Hamiltonian} & C_n \text{ with } \mathcal{H}_A \otimes \mathcal{H}_B \arrow[u, "\substack{\text{replica trick}\\+\\\text{discretization}}"] \arrow[r, "\substack{\text{[Casini, Huerta; 1406.2991];}\\\text{[Radičević; 1605.09396];}\\\text{[Moitra, Soni, Trivedi; 1811.06986]}}"'] & C_n^\star \text{ superselection sectors} \arrow[u, "\substack{\text{[Lin, Radičević; 1808.05939];}\\\text{[AB, Panero; 2404.01987]}}"']
\end{tikzcd}$$

- Consider Abelian gauge theories on the lattice.
- In a basis that diagonalizes the electric field operators we have a well defined notion of electric flux on every link.
- Given a bipartition of the lattice we can define the net electric flux through the boundary $k = \sum_a k_a$.
- $\mathcal{H} = \bigoplus_k \mathcal{H}_A^{(k)} \otimes \mathcal{H}_B^{(-k)}$

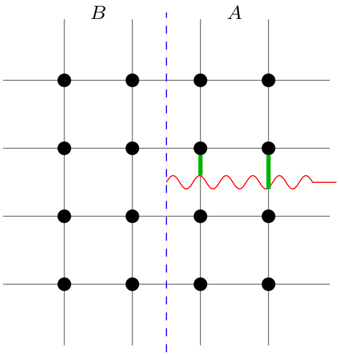


- It was shown in [Buividovich, Polikarpov; 0806.3376], [Casini, Huerta, Rosabal; 1312.1183] that the entanglement entropy of an Abelian gauge theory admits the following decomposition

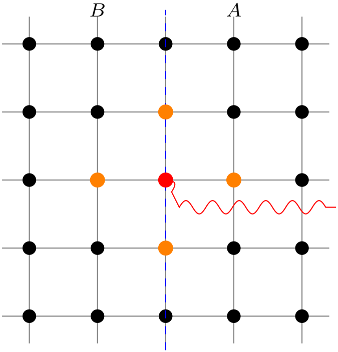
$$S_A = -\sum_k p^{(k)} \log p^{(k)} - \sum_k p^{(k)} S(\rho_A^{(k)}).$$

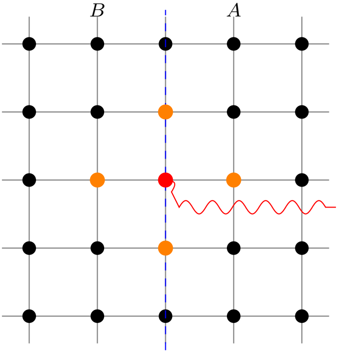
- The first term depends on the classical probability distribution of the flux through the boundary.
- There are arguments suggesting that in the continuum limit the universal part of S_A takes contributions only from the second, distillable term [Casini, Huerta; 1406.2991], [Moitra, Soni, Trivedi; 1811.06986].
- Still, the non-factorization of $\mathcal{H}$ makes the definition of a replica geometry less clear.

$\mathbb{Z}_N$ spin model in $2D$

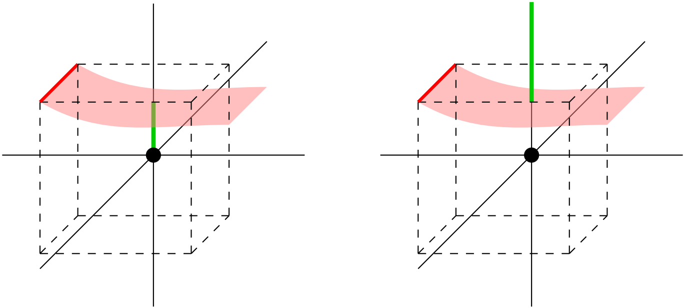


$\mathbb{Z}_N$ spin model in $2D$





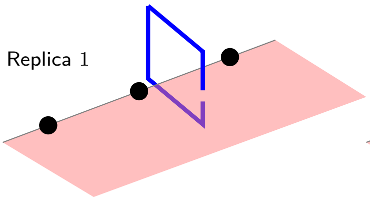
Abelian gauge theories in $3D$

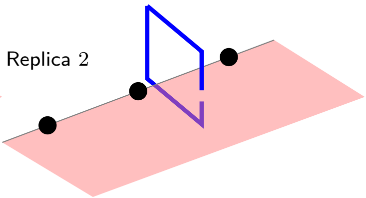


- In a system of n replicas gauge fields on the boundary belong to $4n$ plaquettes.
- Notice that this geometry constraints gauge transformations along the entangling surface to be the same on all replicas.

Abelian gauge theories in $3D$: central-plaquette geometry

- Geometrically, an other possibility is to locate the entangling surface along the links of the spin model.
- In this geometry spins along the entangling surface have an enlarged number of nearest neighbors.





- This "central-plaquette" geometry was introduced in [Chen et al., 1503.01766].

Boundary conditions

- In general the topology of the lattice manifold has an effect on the duality transformation.
- For example, in the $2D$ Ising model on a torus

$$Z(\beta) \propto Z^{\star}_{pp}(\beta^{\star}) + Z^{\star}_{pa}(\beta^{\star}) + Z^{\star}_{ap}(\beta^{\star}) + Z^{\star}_{aa}(\beta^{\star}).$$