Loschmidt echo, emerging dual unitarity and scaling of generalized temporal entropies after quenches to the critical point

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QuantFunc Valencia 09/2024

Motivation

Quantum many-body dynamics is hard…

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…but doesn’t have to be *exponentially* hard all the time…

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Quantum many-body dynamics is hard…

…but doesn’t have to be *exponentially* hard all the time…

…and it’s always nice if we can see that analytically!

Motivation

New methods for transverse contraction of tensor networks associated with time evolution of quantum many-body systems

New concepts: transition matrices and generalized entropies

Can field theory predict help us predict their behavior?

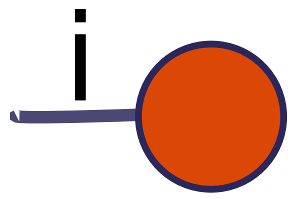
arXiv: 2405.14706

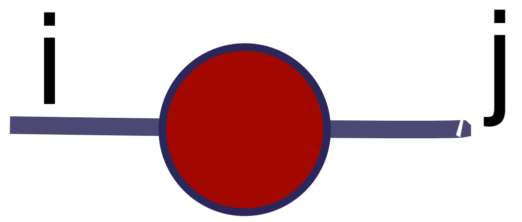
with Luca Tagliacozzo (CSIC)

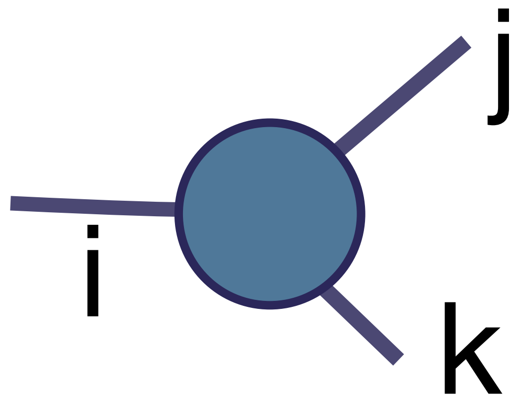
Roadmap

- Many-body dynamics using tensor networks, techniques and barriers
- Can conformal field theories help us?
- CFT predictions for Loschmidt echo

Quick dictionary: Tensor Notation

- Vector v^i 

- Matrix M^{ij} 

- Rank-3 Tensor T^{ijk} 

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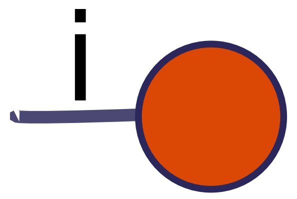
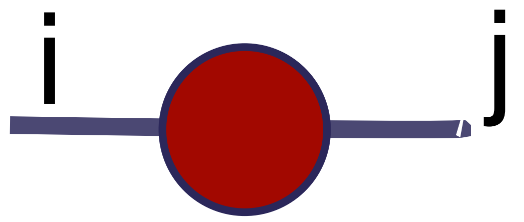
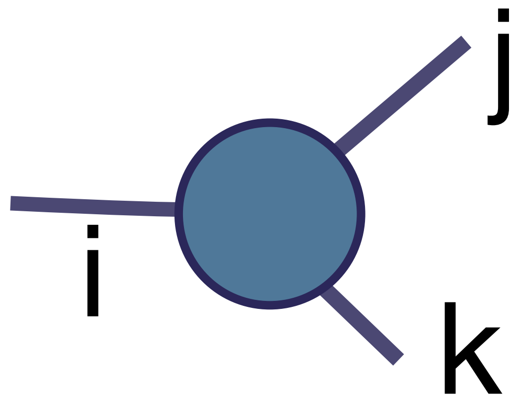
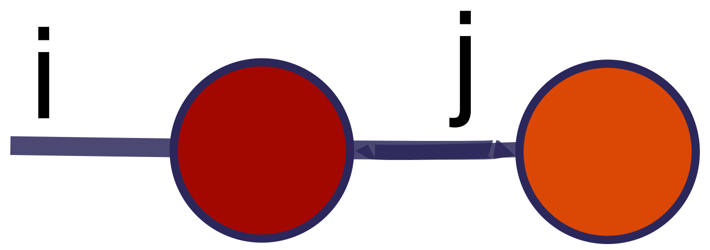
- Vector v^i 

- Matrix M^{ij} 

- Rank-3 Tensor T^{ijk} 

- Matrix-Vector product $M^{ij}v^j$ 

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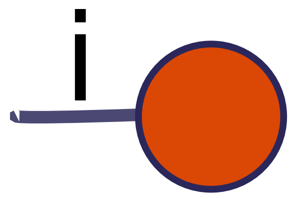

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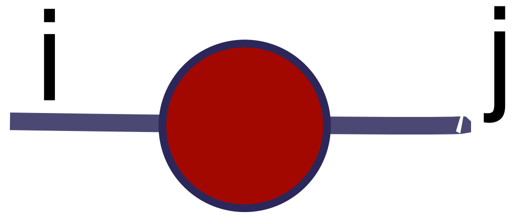

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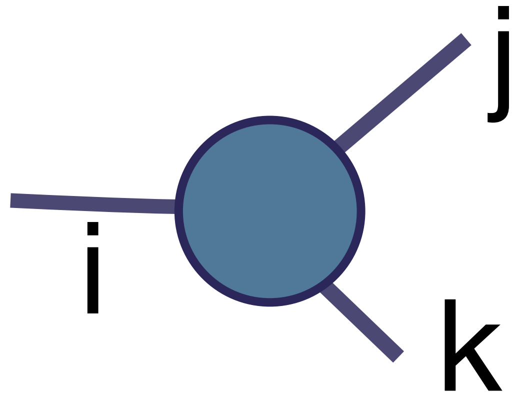

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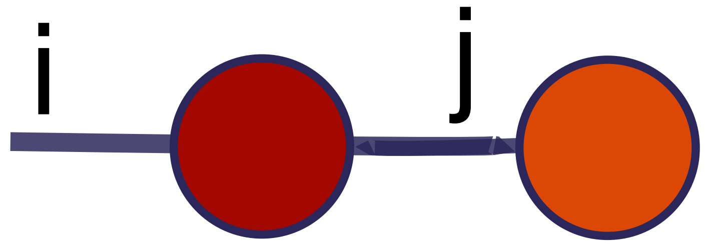
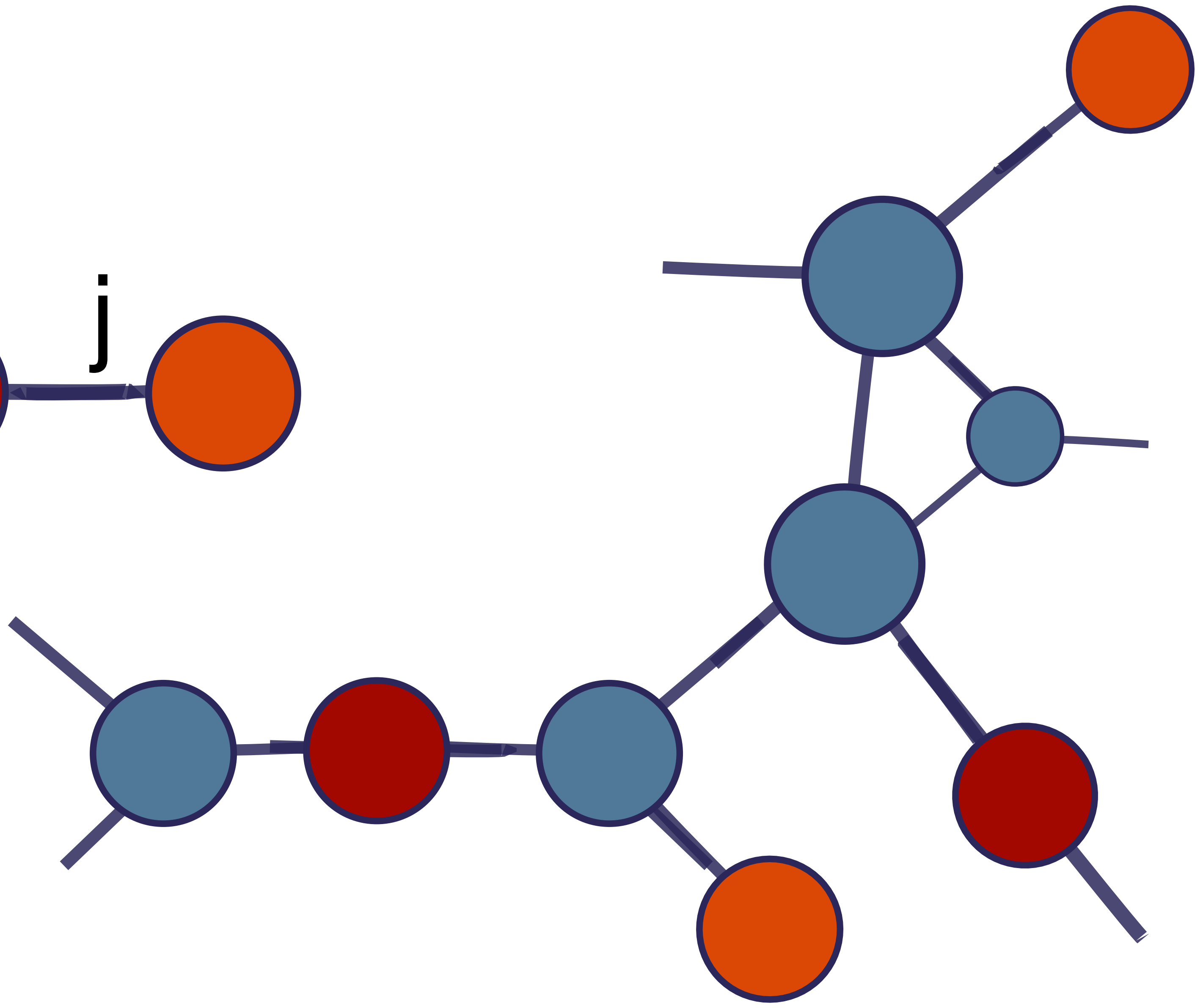


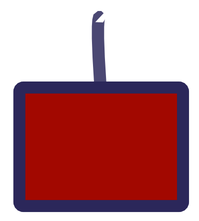
- Arbitrarily complicated contraction



Quick dictionary: Matrix Product States

- Quantum many-body system wave function

$$|\psi\rangle = \sum_i c_i |i\rangle$$



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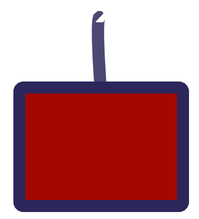

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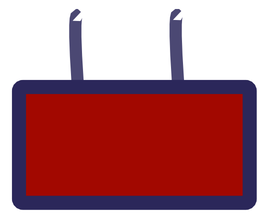

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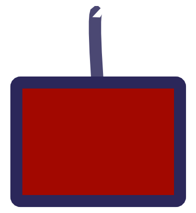

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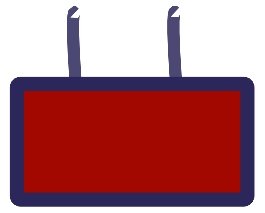

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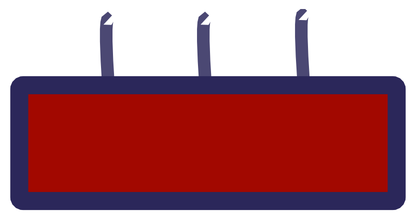

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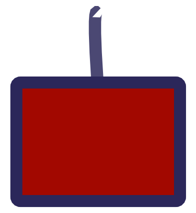

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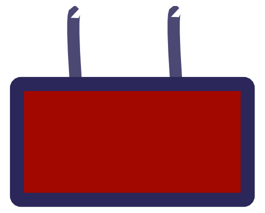

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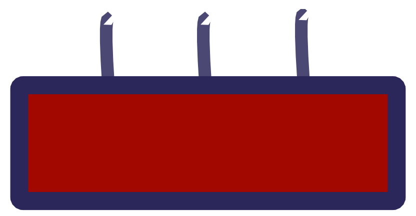
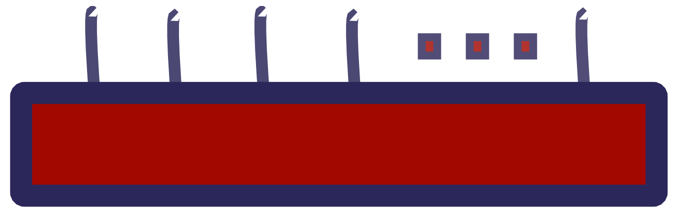


$$|\psi\rangle = \sum_{i_1..i_N} c_{i_1..i_N} |i_1 i_2 .. i_N\rangle$$



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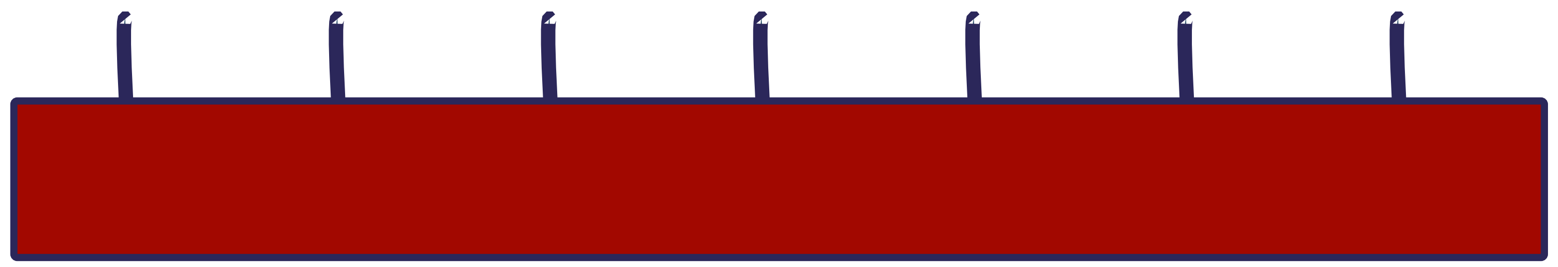
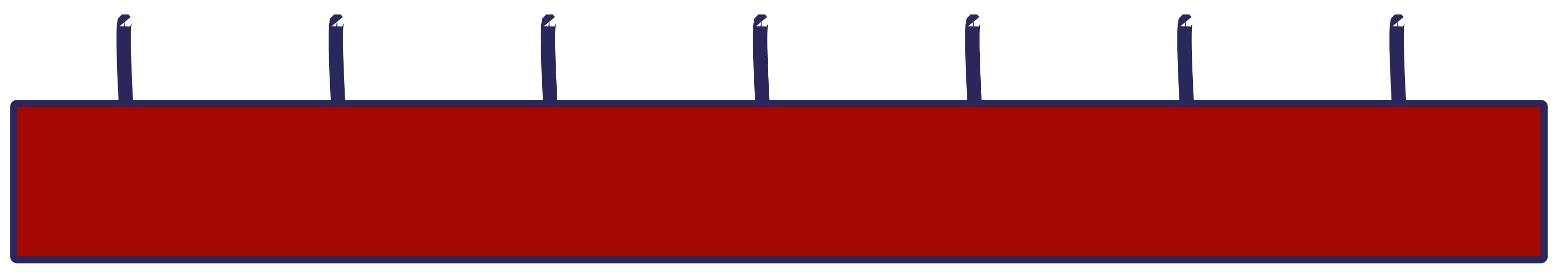
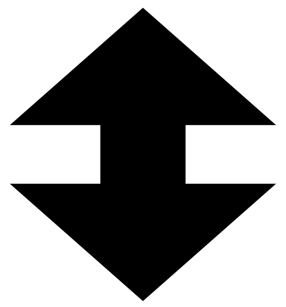


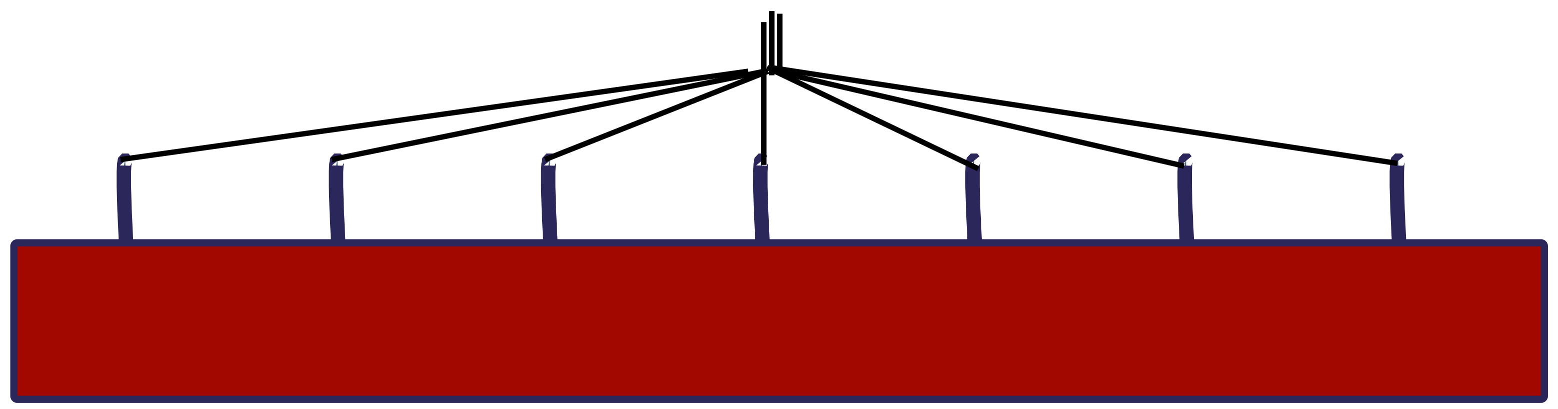
$$c_{i_1 i_2..i_N}$$

~exp(N) parameters



Rank-n tensor





very big vector

Quick dictionary: Matrix Product States

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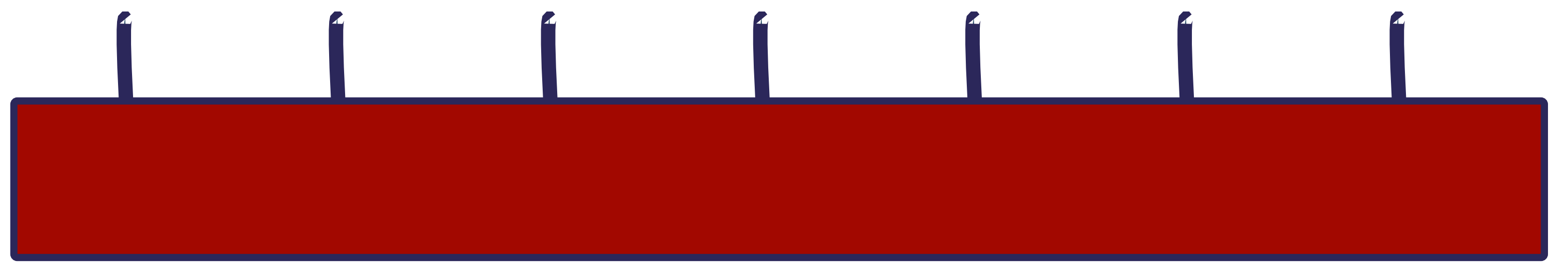
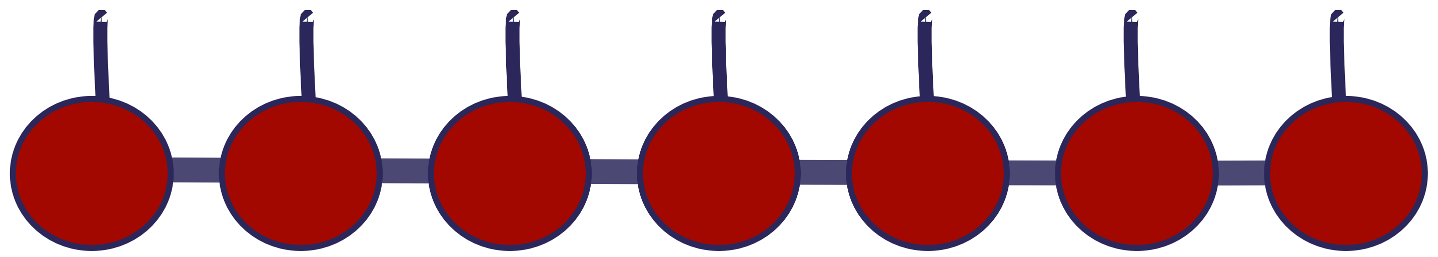


$$c_{i_1 i_2..i_N}$$

~exp(N) parameters

Compressed representation for a 1D system:
Matrix Product State



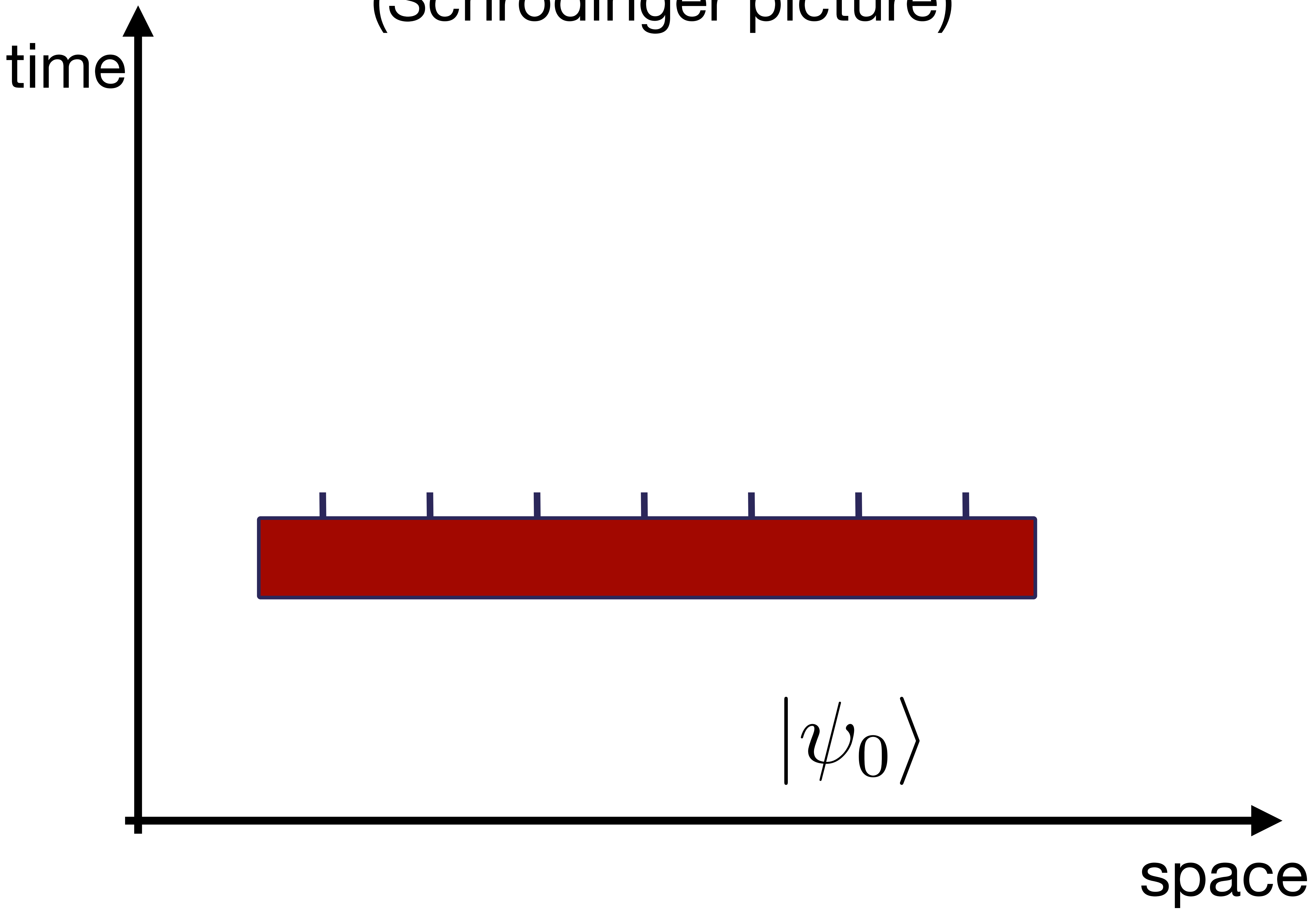
$$\Rightarrow \quad [A^1]^{\alpha}_{i_1}[A^2]^{\alpha\beta}_{i_2}...[A^{N-1}]^{\upsilon\zeta}_{i_{N-1}}[A^N]^{\zeta}_{i_N}$$

~ poly(N) parameters

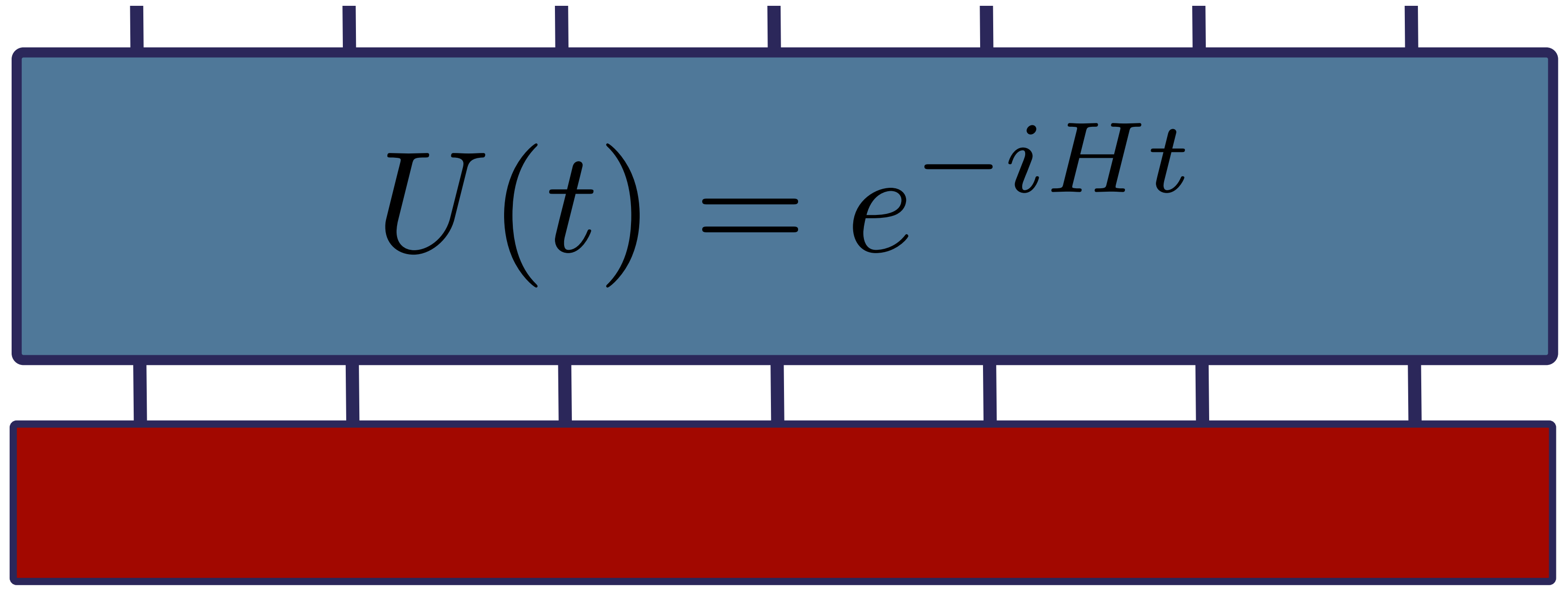
- In principle: can be as faithful as necessary
  - In practice: useful only for states with low entanglement (area law…)

Time evolution

Time evolution for a one-dimensional system
(Schrödinger picture)

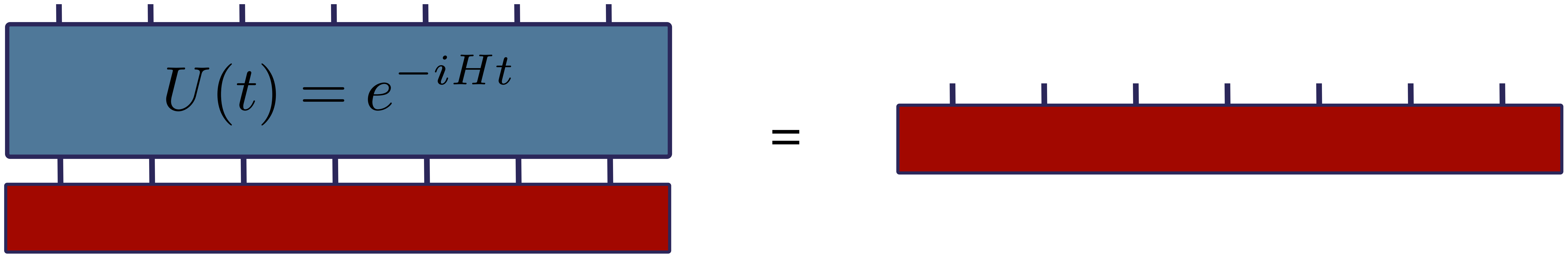


Time evolution



$$U(t)|\psi_0\rangle$$

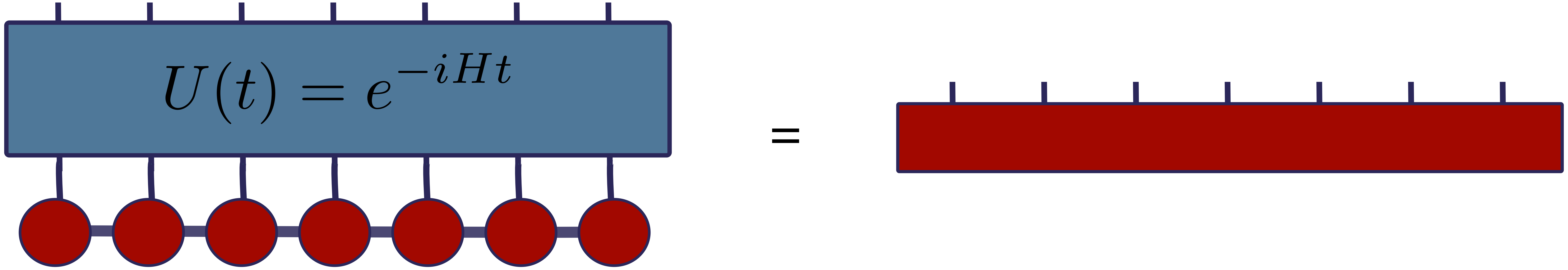
Time evolution



$$U(t)|\psi_0\rangle \quad = \quad |\psi(t)\rangle$$

Time evolution

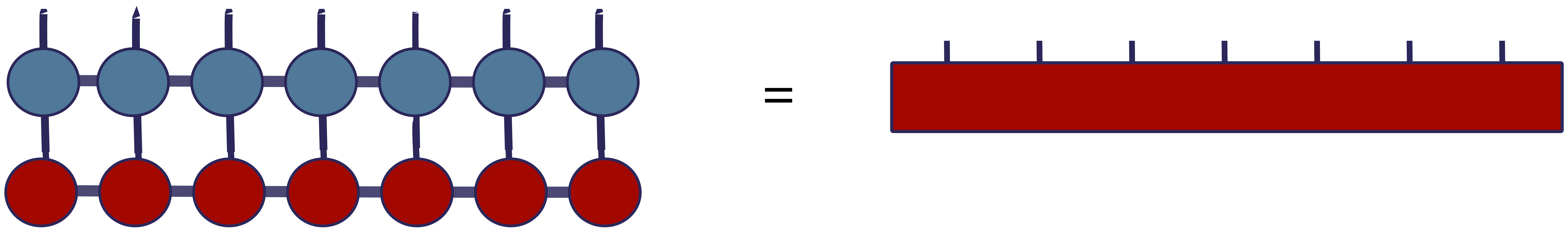
MPS



$U(t)|\psi_0\rangle$ = $|\psi(t)\rangle$

Time evolution

MPO
(Matrix product
operator)



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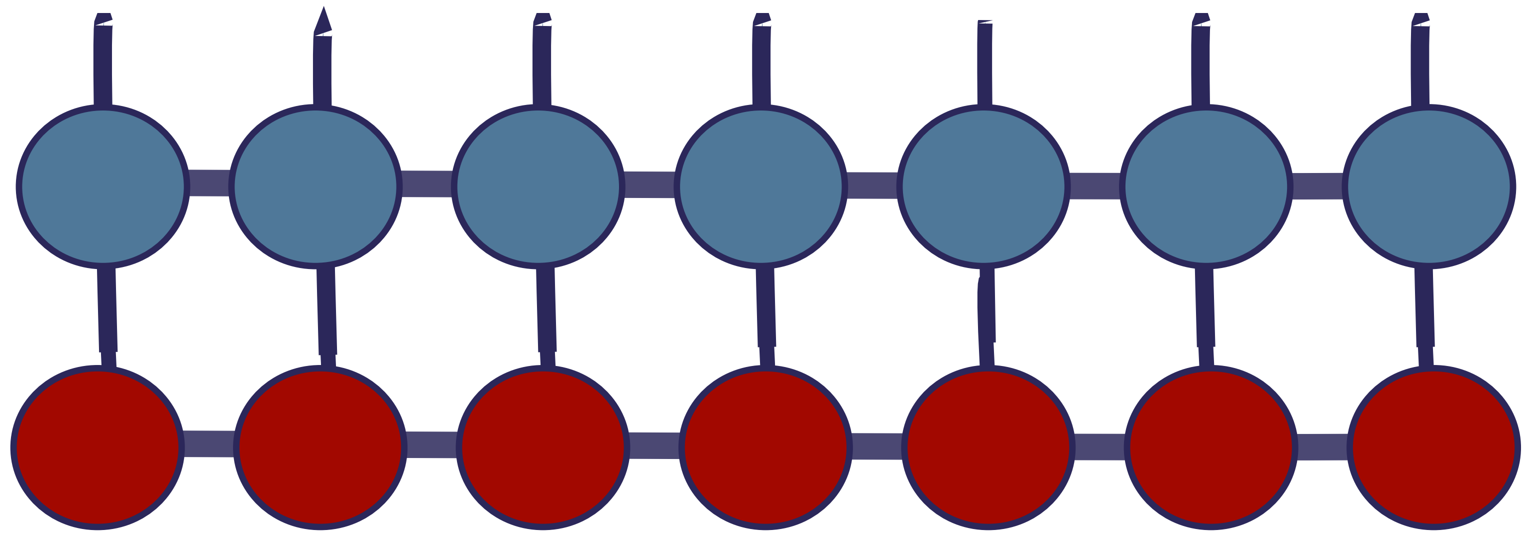
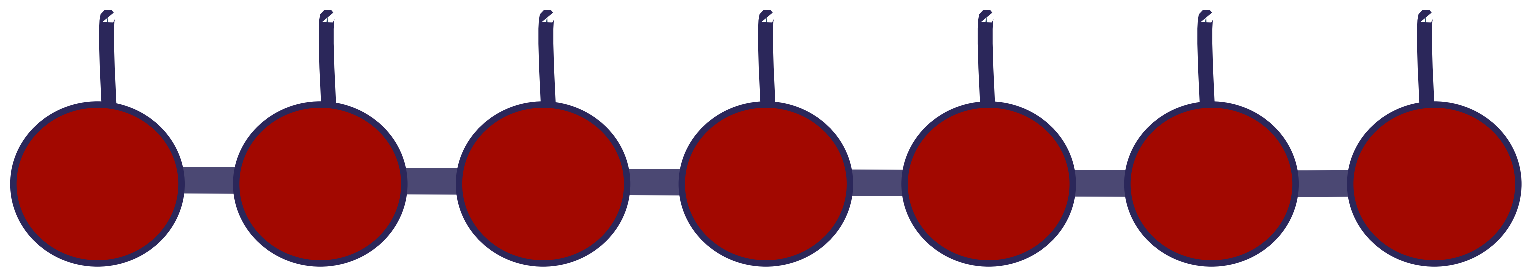
Time evolution



$\approx$



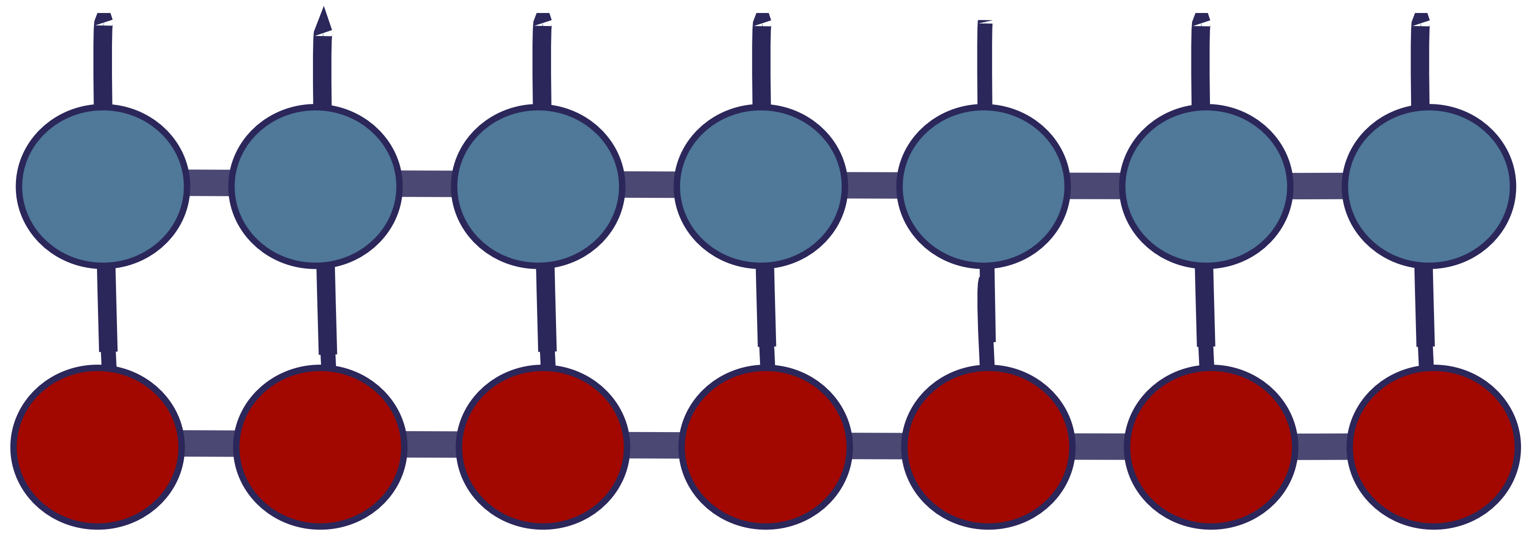
MPS

$$U(t)|\psi_0\rangle \quad = \quad |\psi(t)\rangle$$

Time evolution

Not efficient: entanglement entropy grows linearly with time => need exponentially large tensors

=> Entanglement barrier



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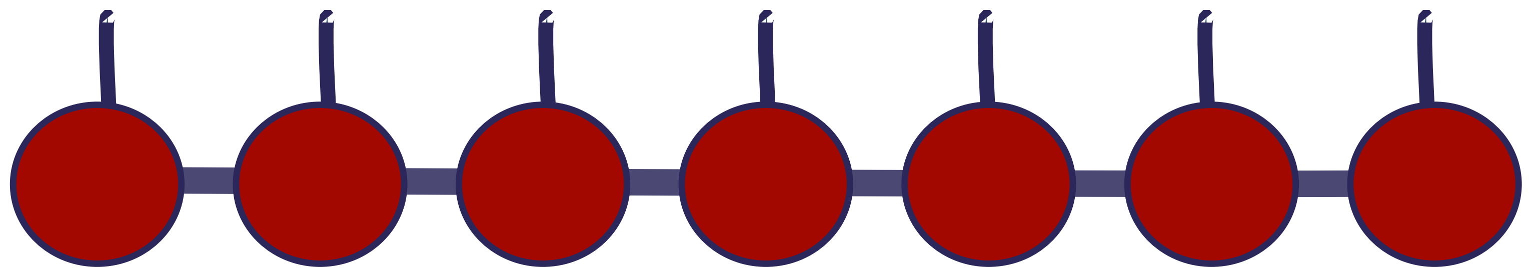

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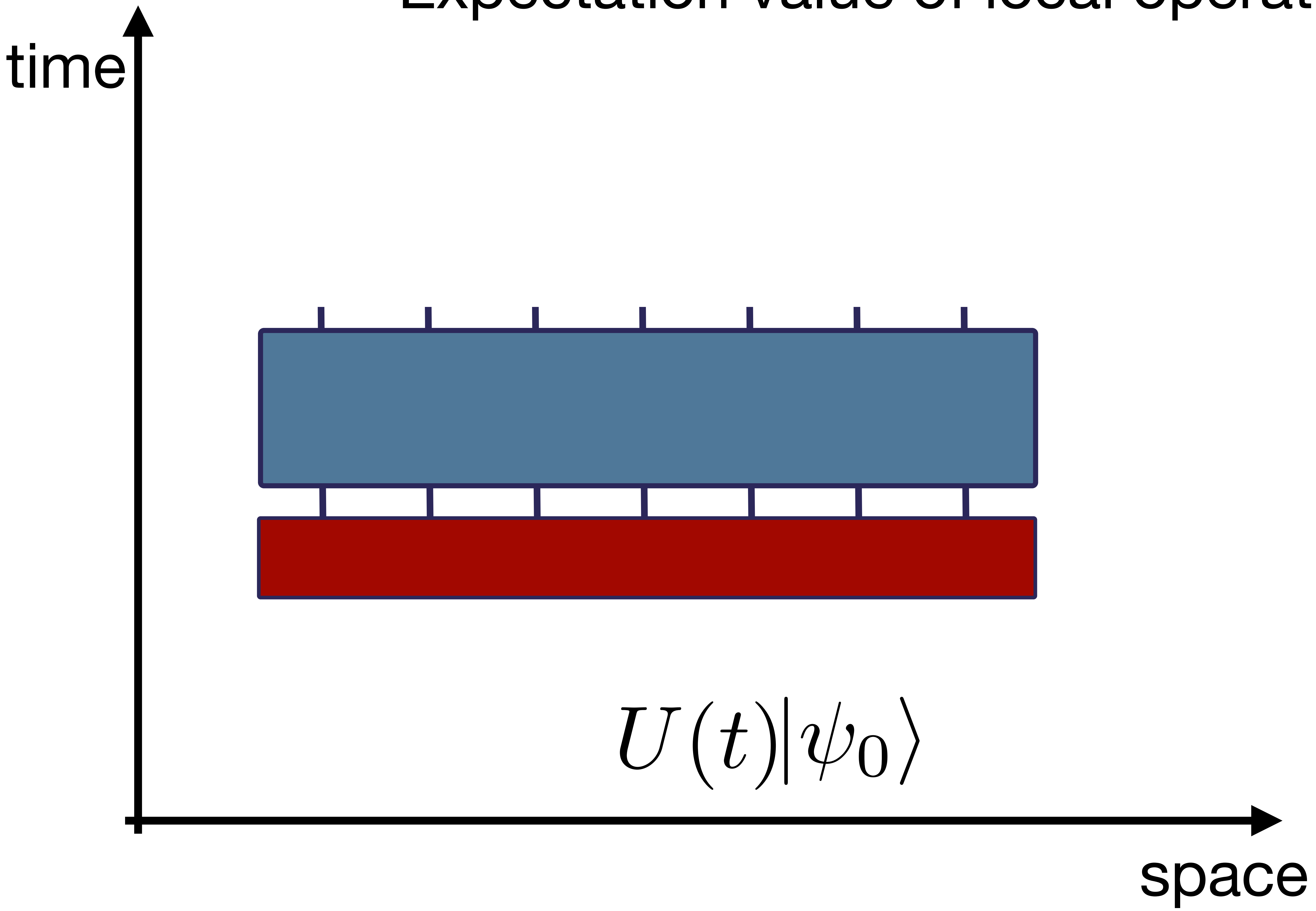
Time evolution and expectation values

Expectation value of local operator



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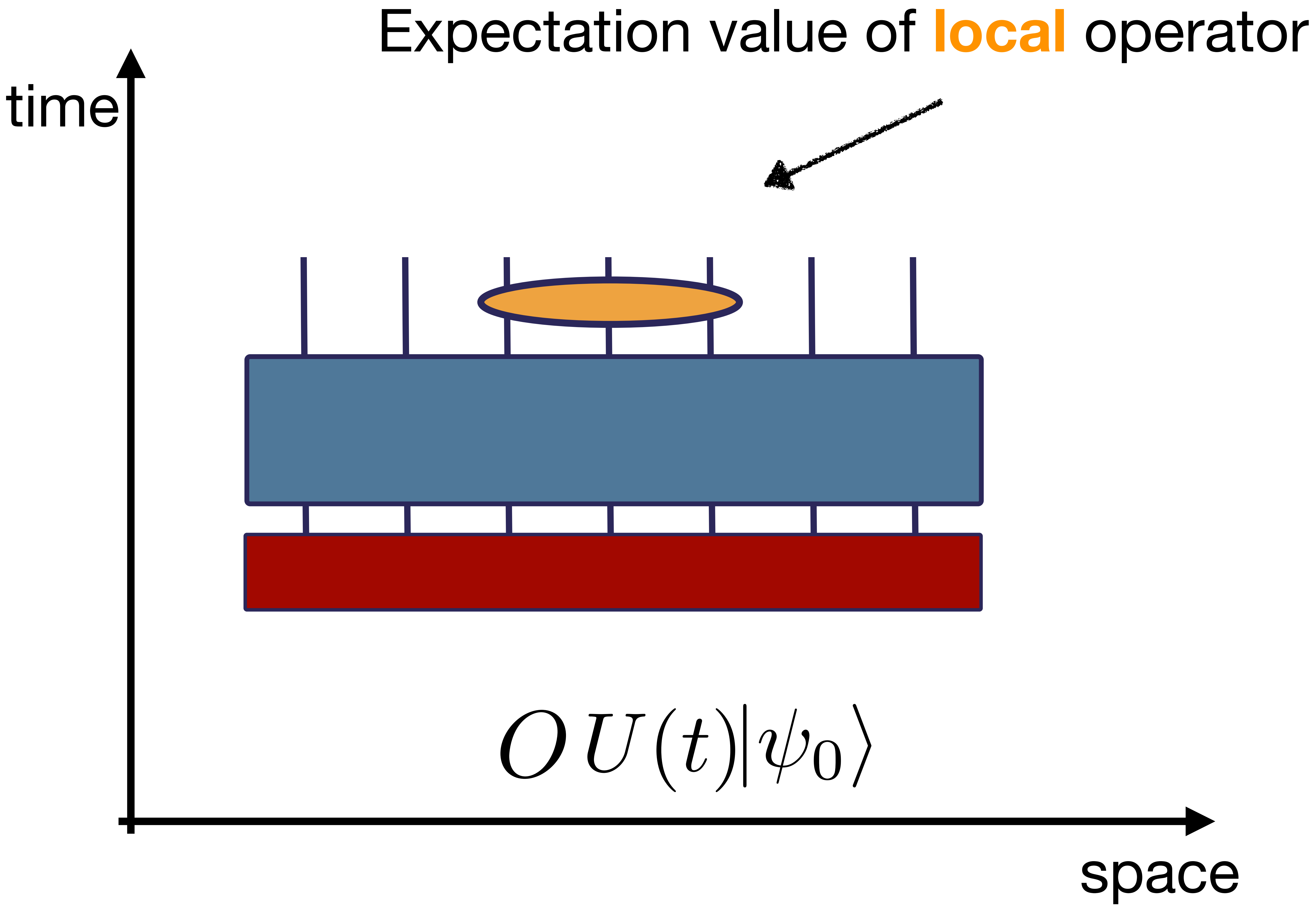

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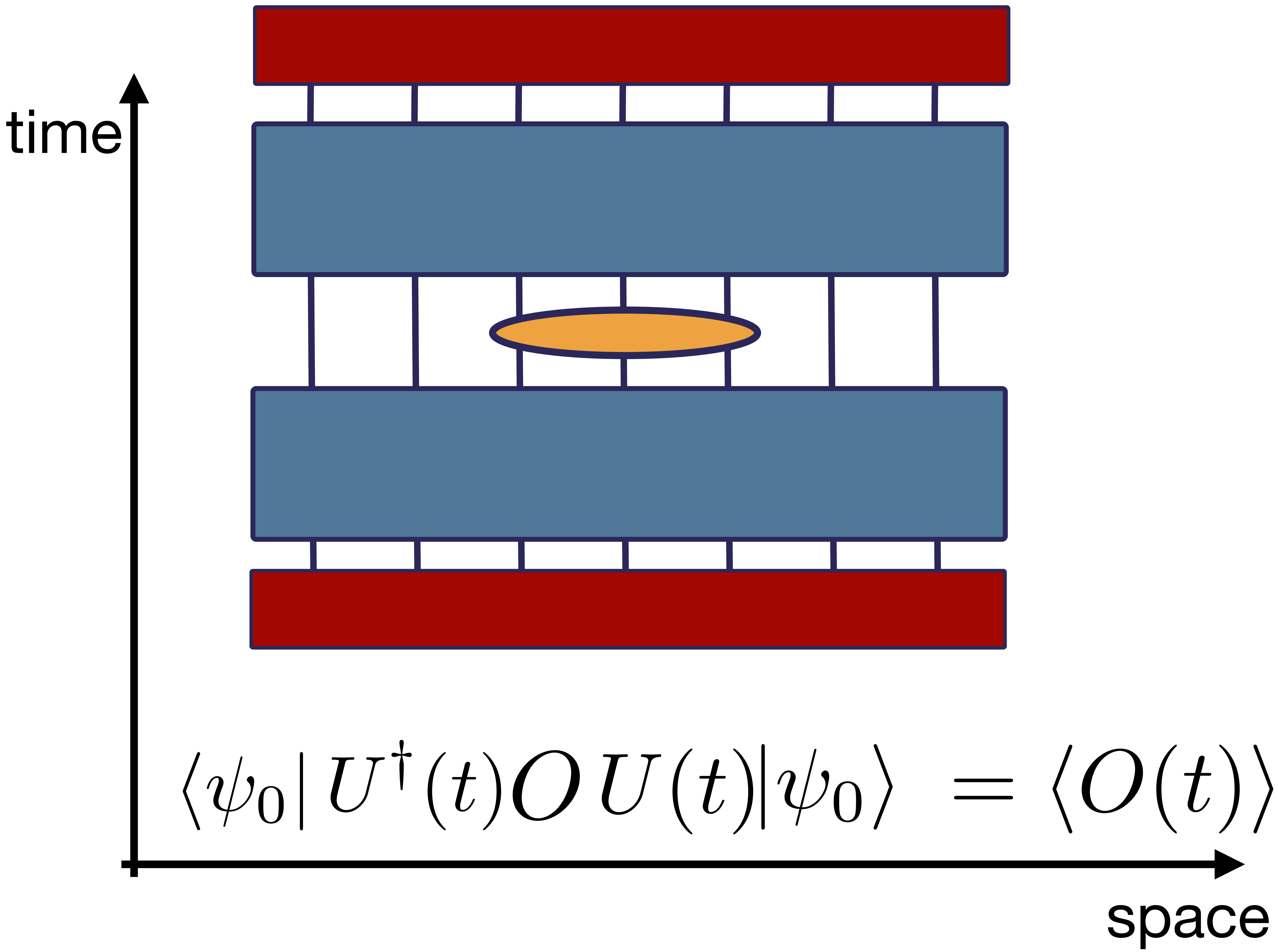
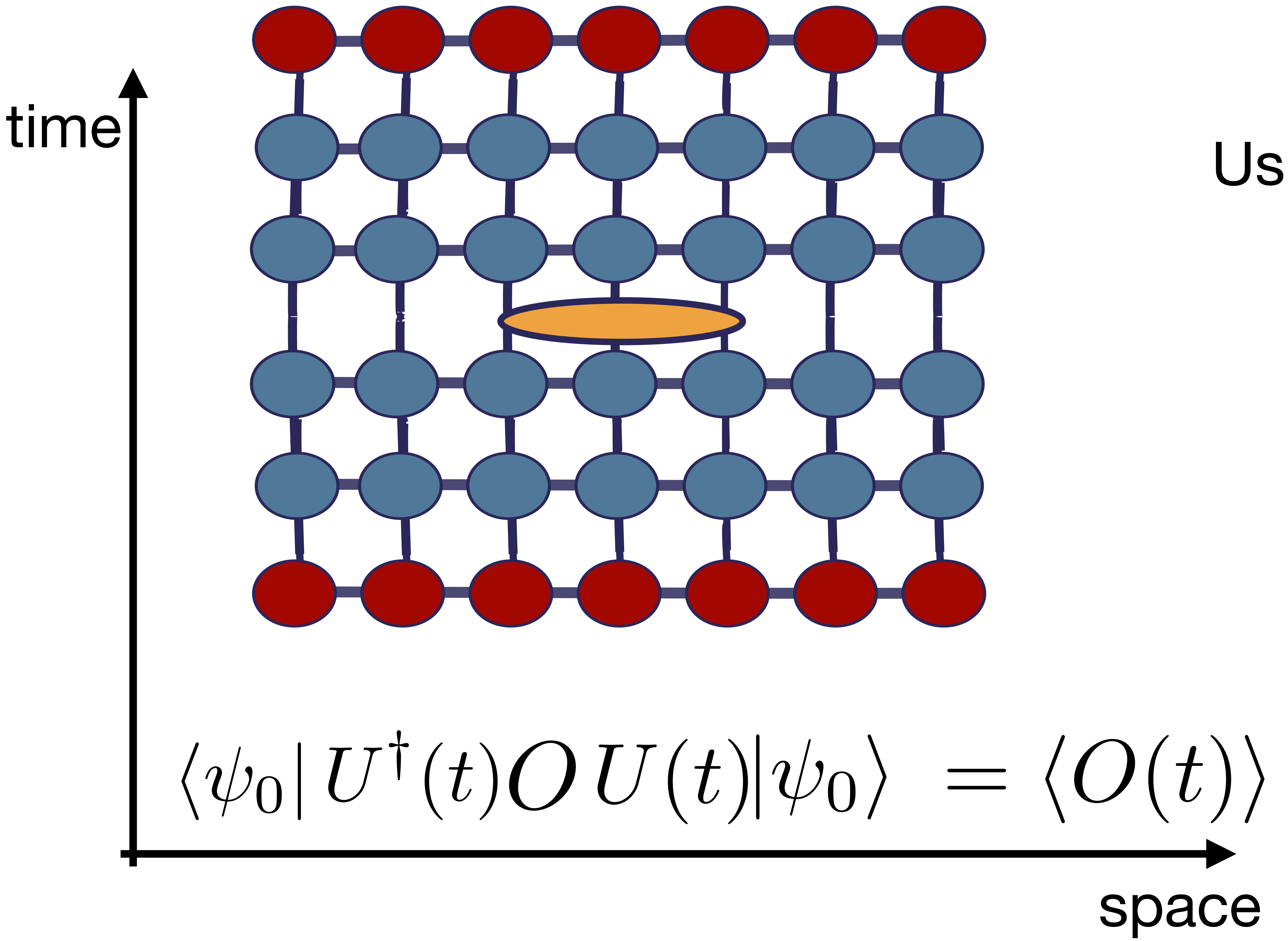


$$\langle\psi_0|\,U^\dagger(t) O\, U(t)|\psi_0\rangle \;=\langle O(t)\rangle$$

Time evolution and expectation values



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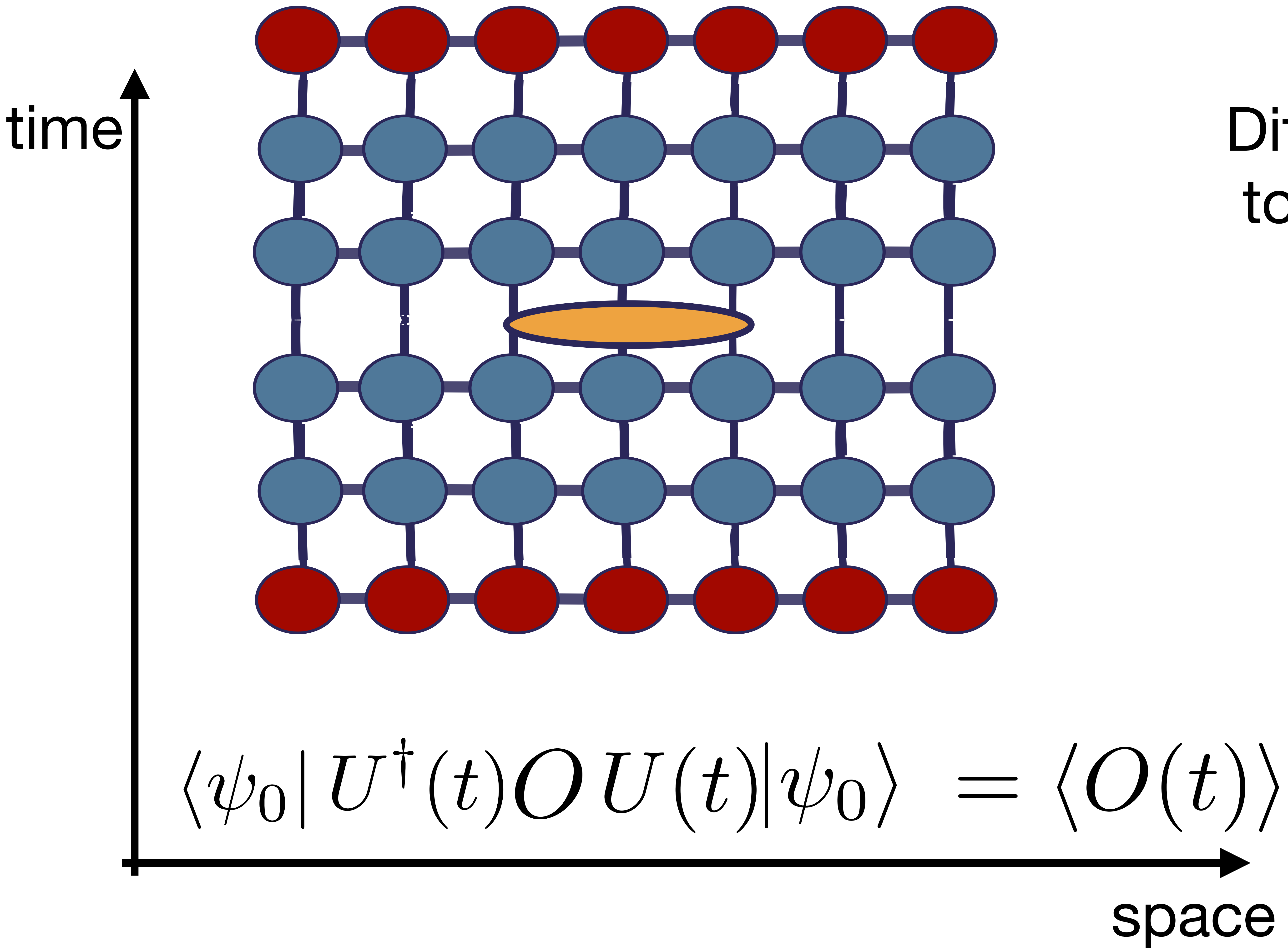
Using MPS/MPO

Time evolution and expectation values



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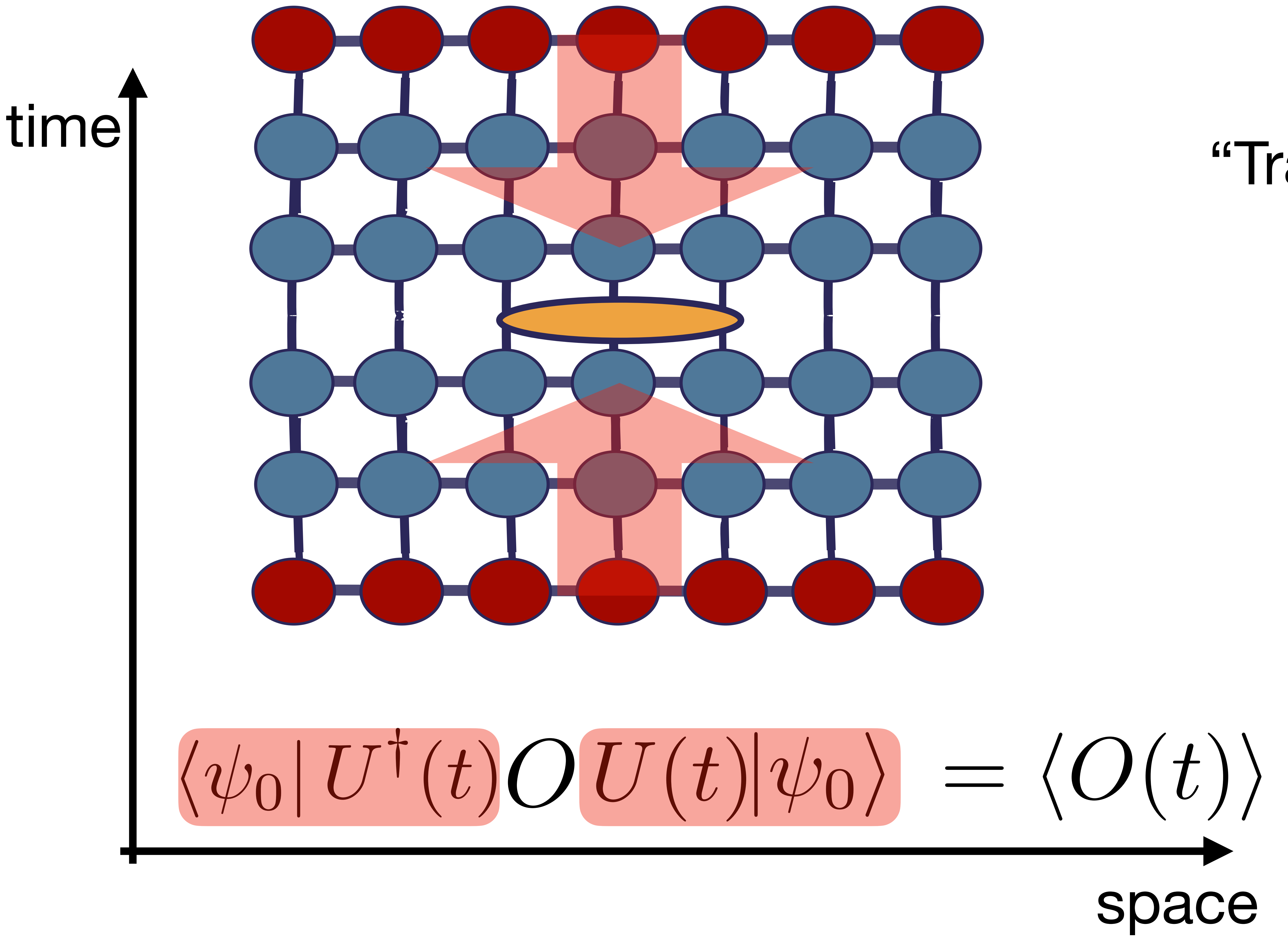
Different possible ways
to contract this TN

Time evolution and expectation values



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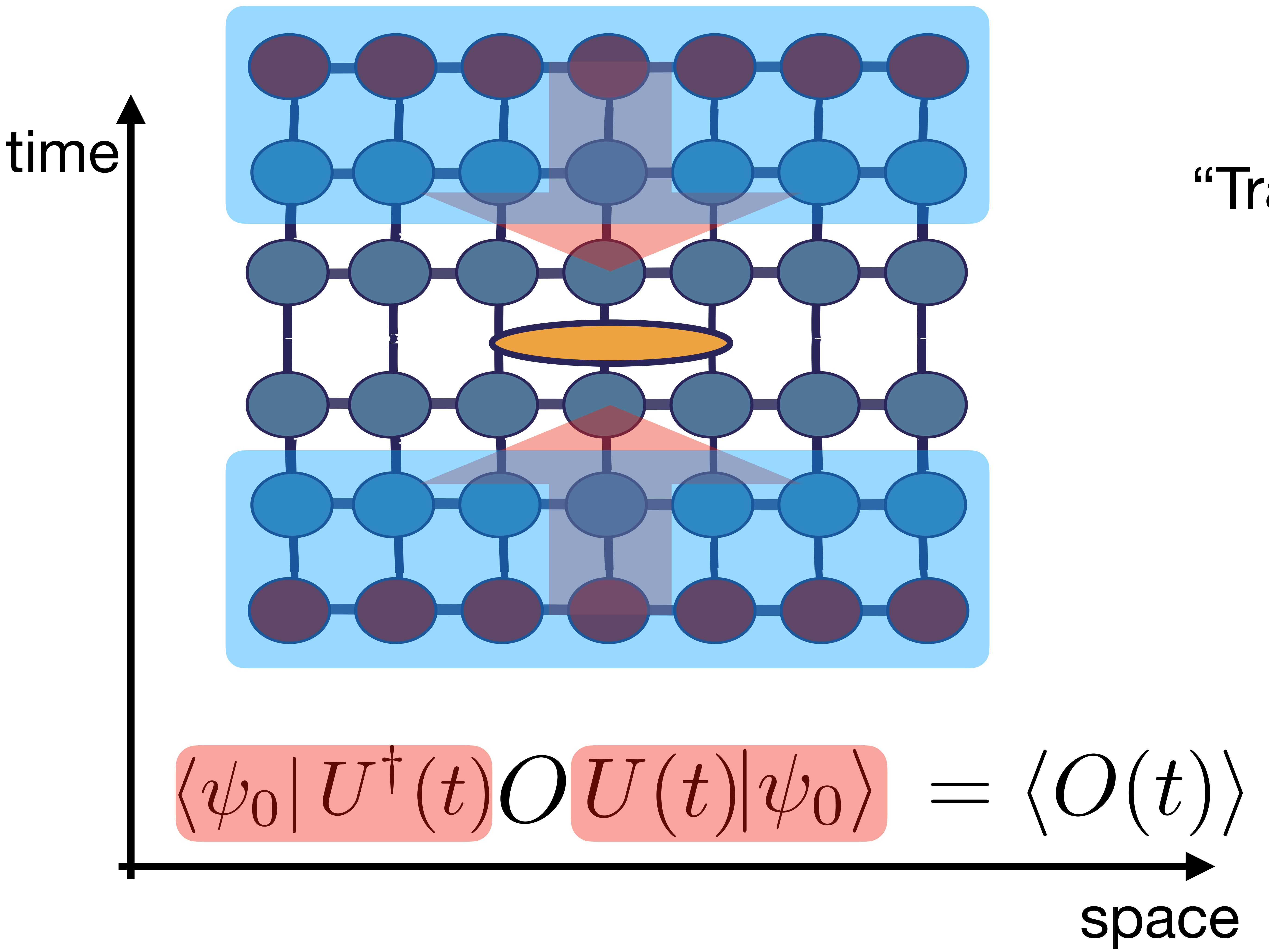
“Traditional” time evolution:

Schrödinger

Time evolution and expectation values



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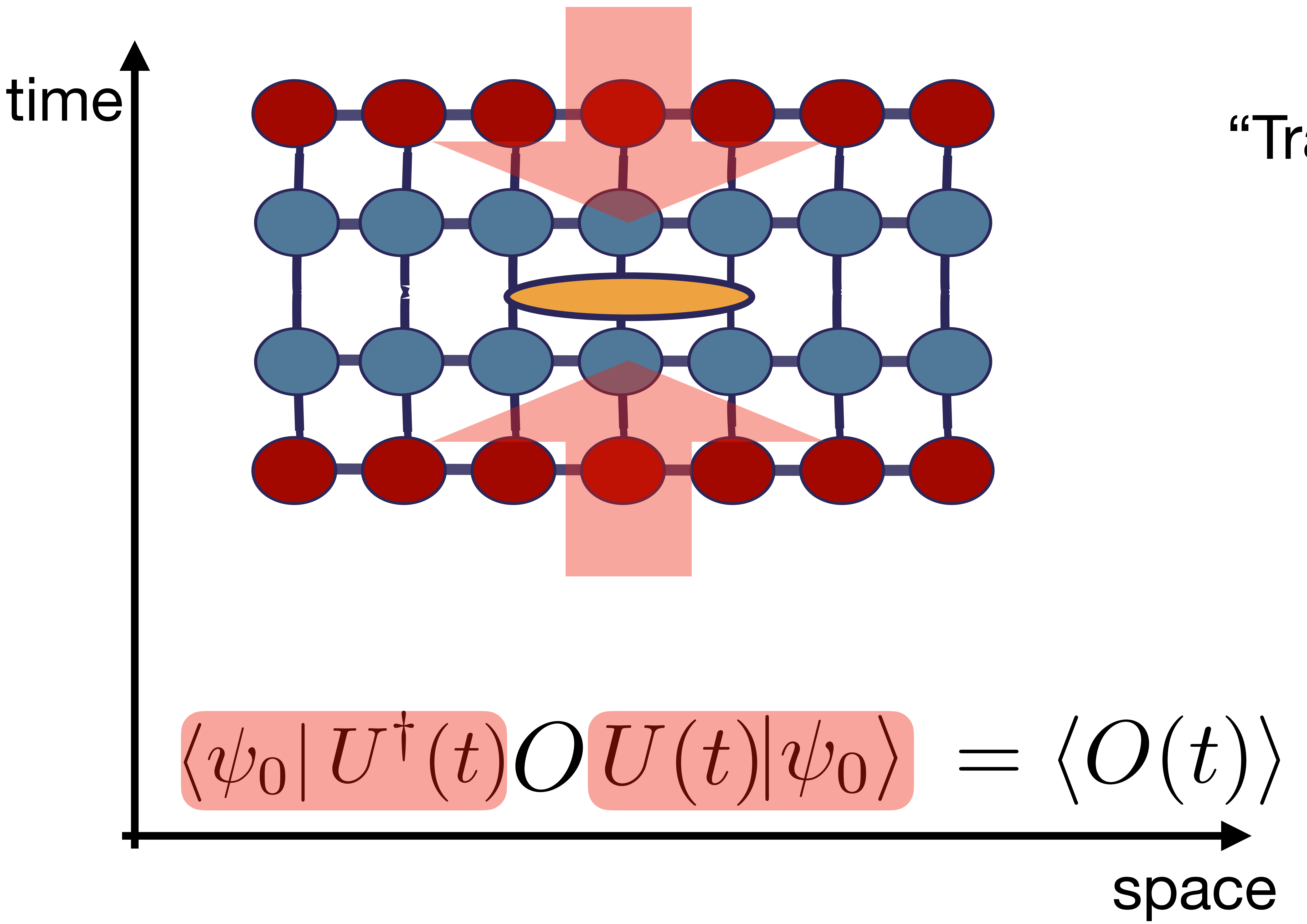

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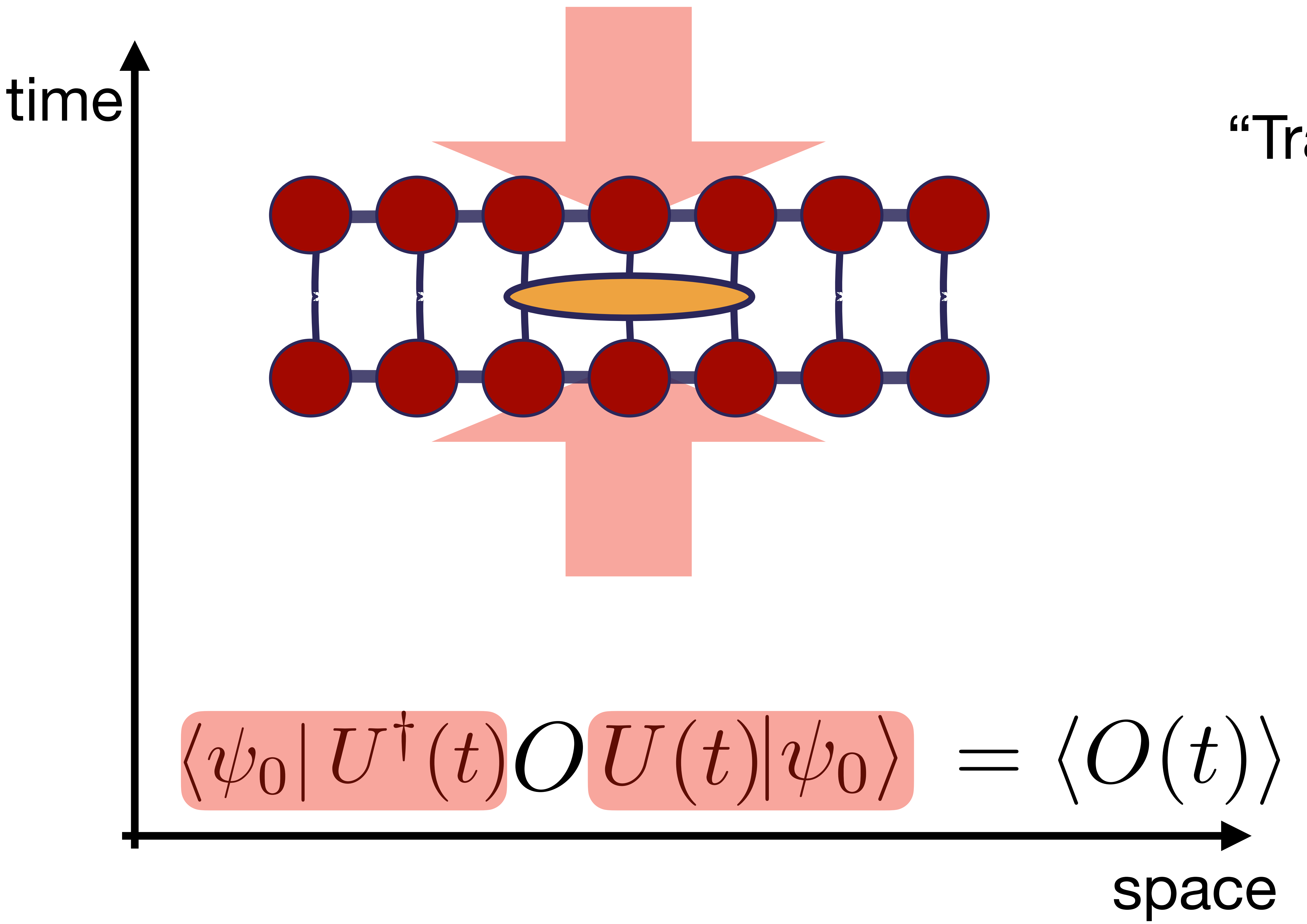

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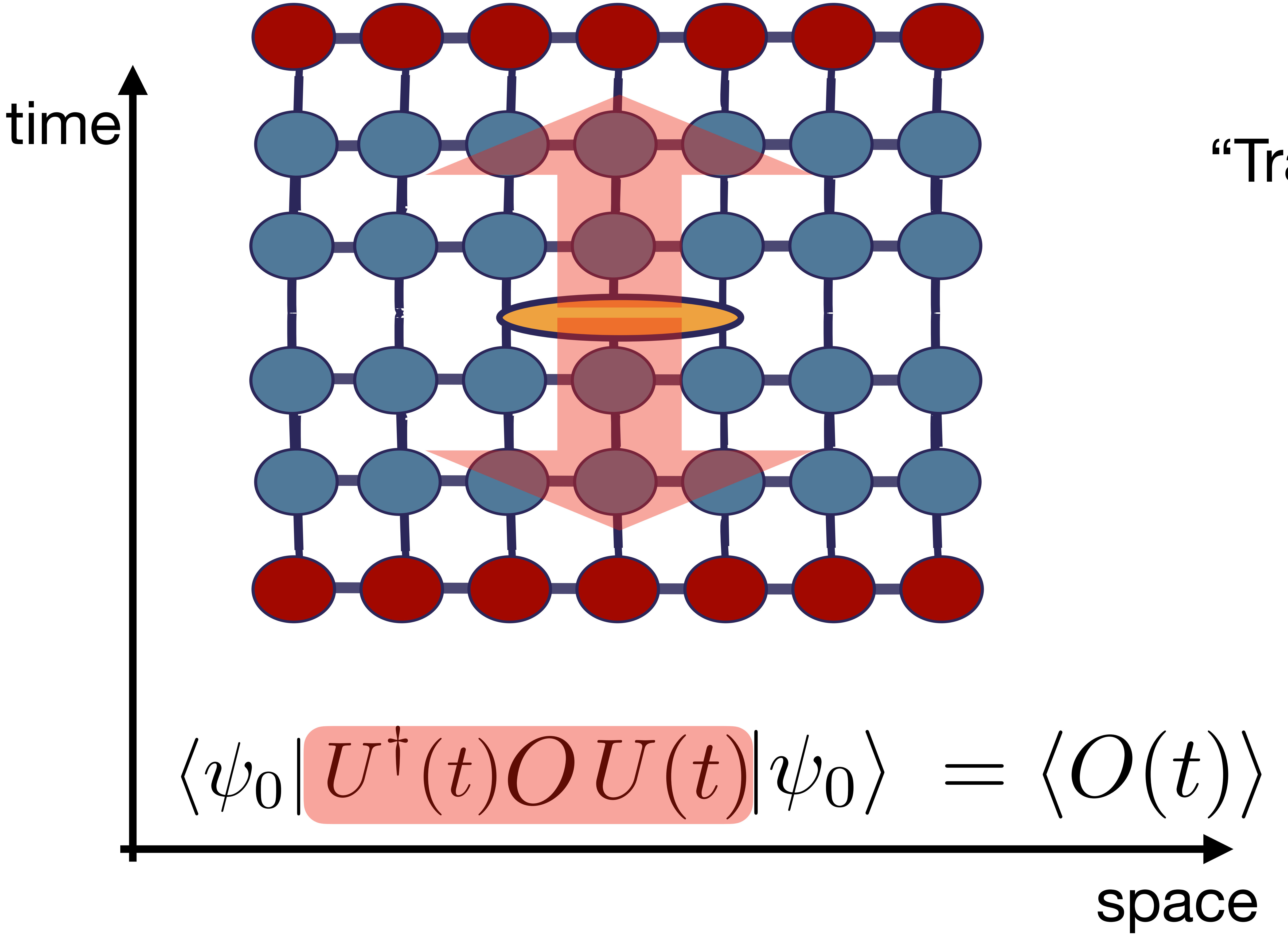
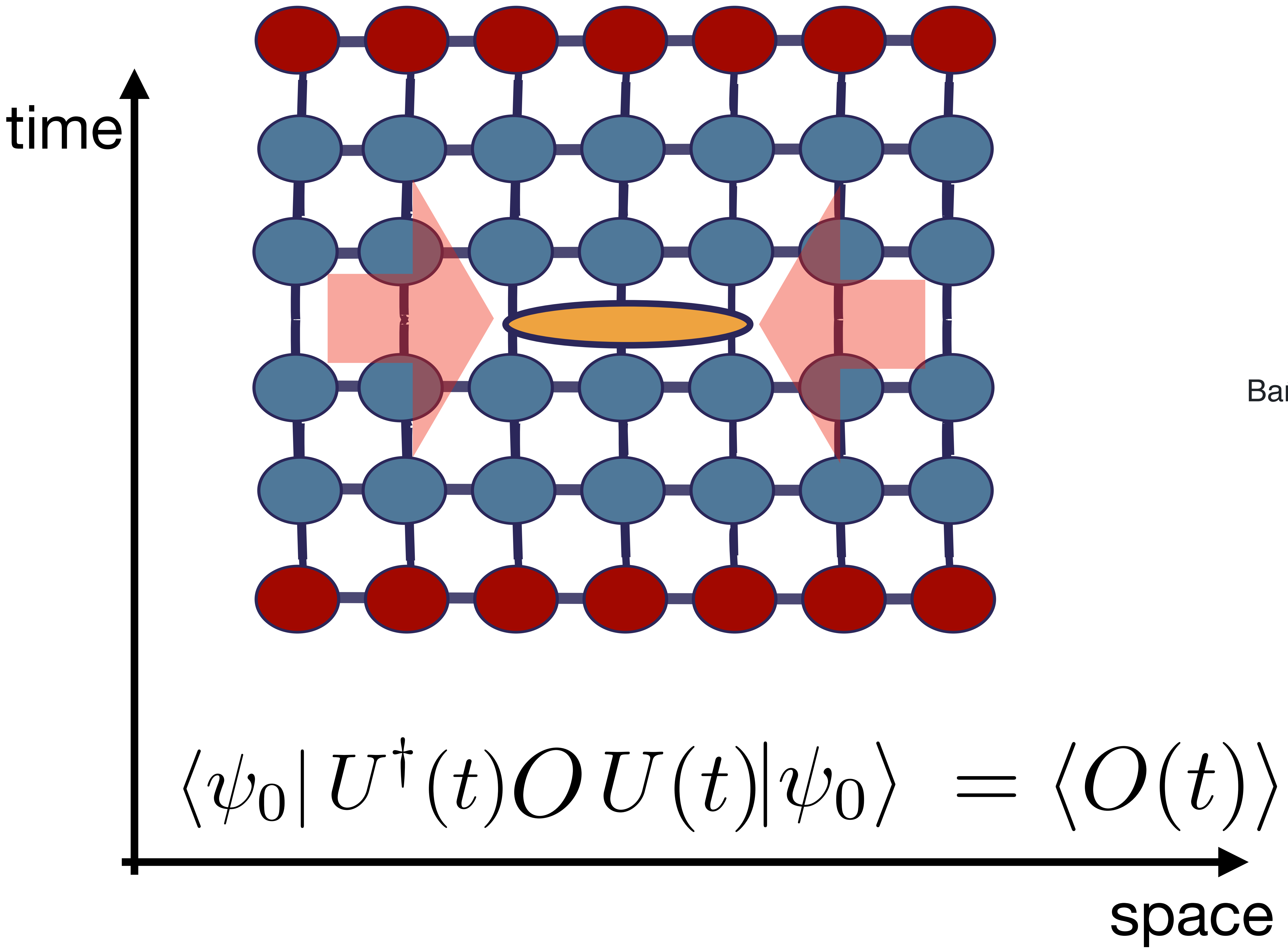


$$\langle\psi_0|U^\dagger(t)\,O\,U(t)|\psi_0\rangle = \langle O(t)\rangle$$

“Traditional” time evolution:

Heisenberg

Time evolution and expectation values



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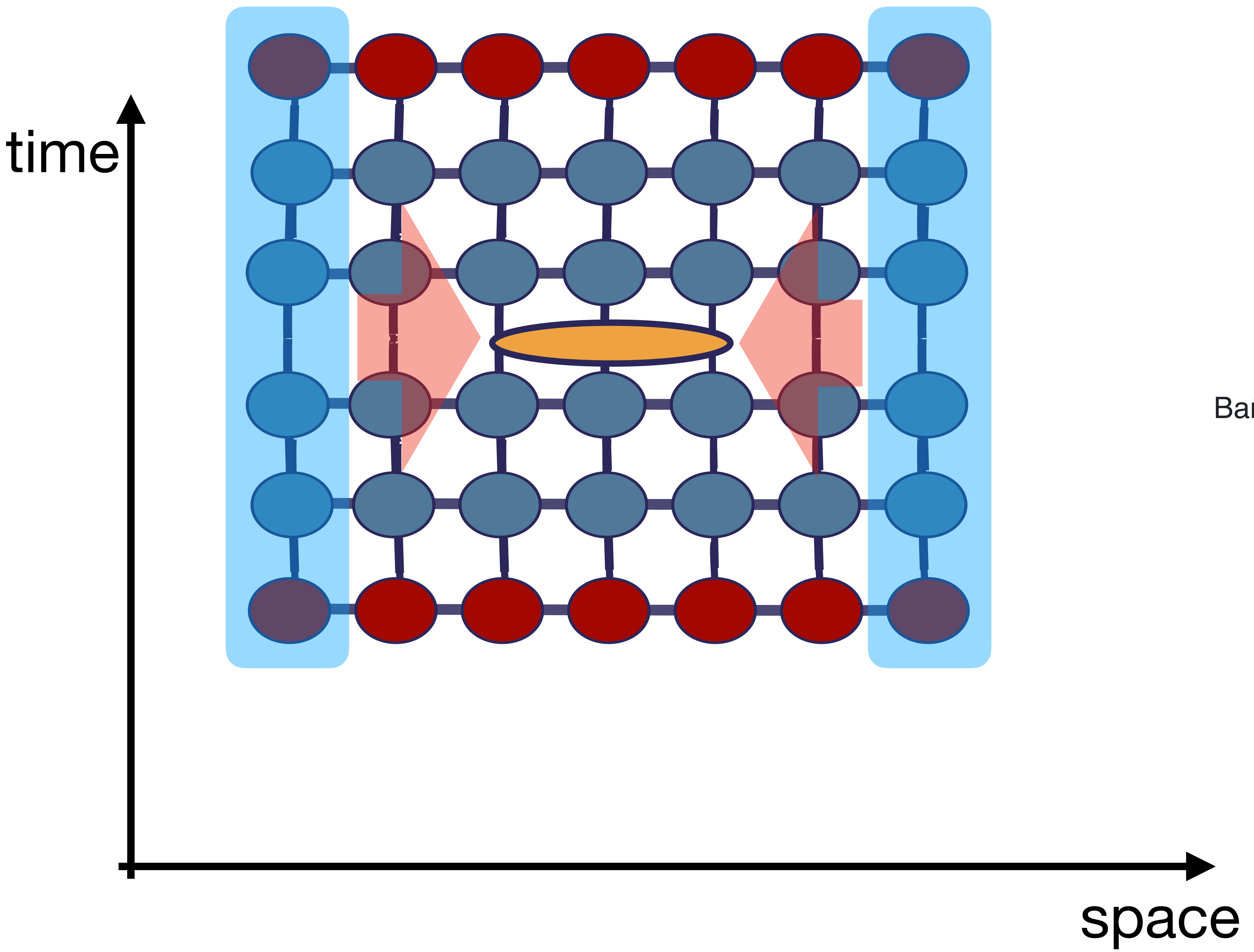
“Transverse” contraction

Bañuls, Hastings, Verstraete, Cirac, PRL 102 240603 (2009)

Transverse contraction



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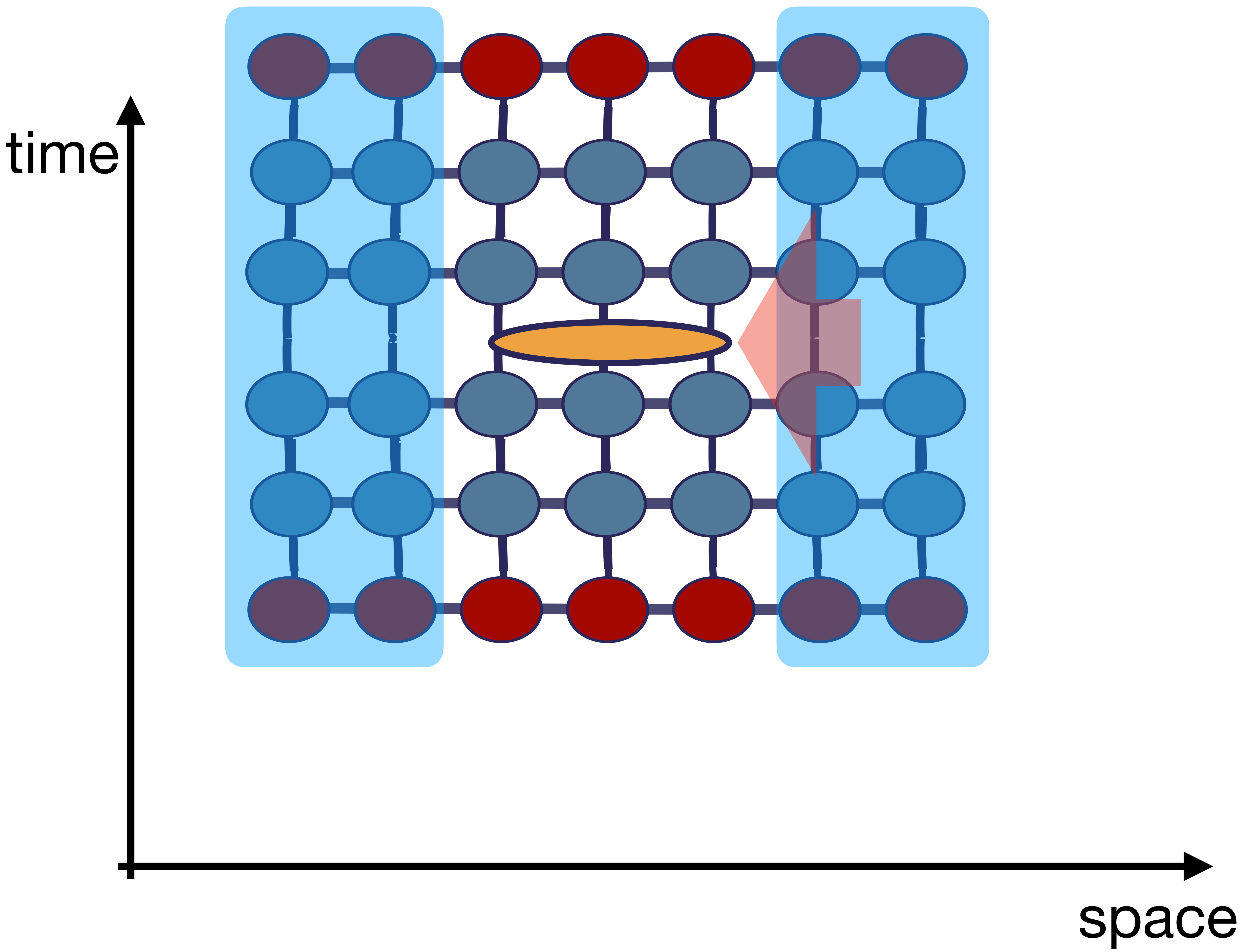
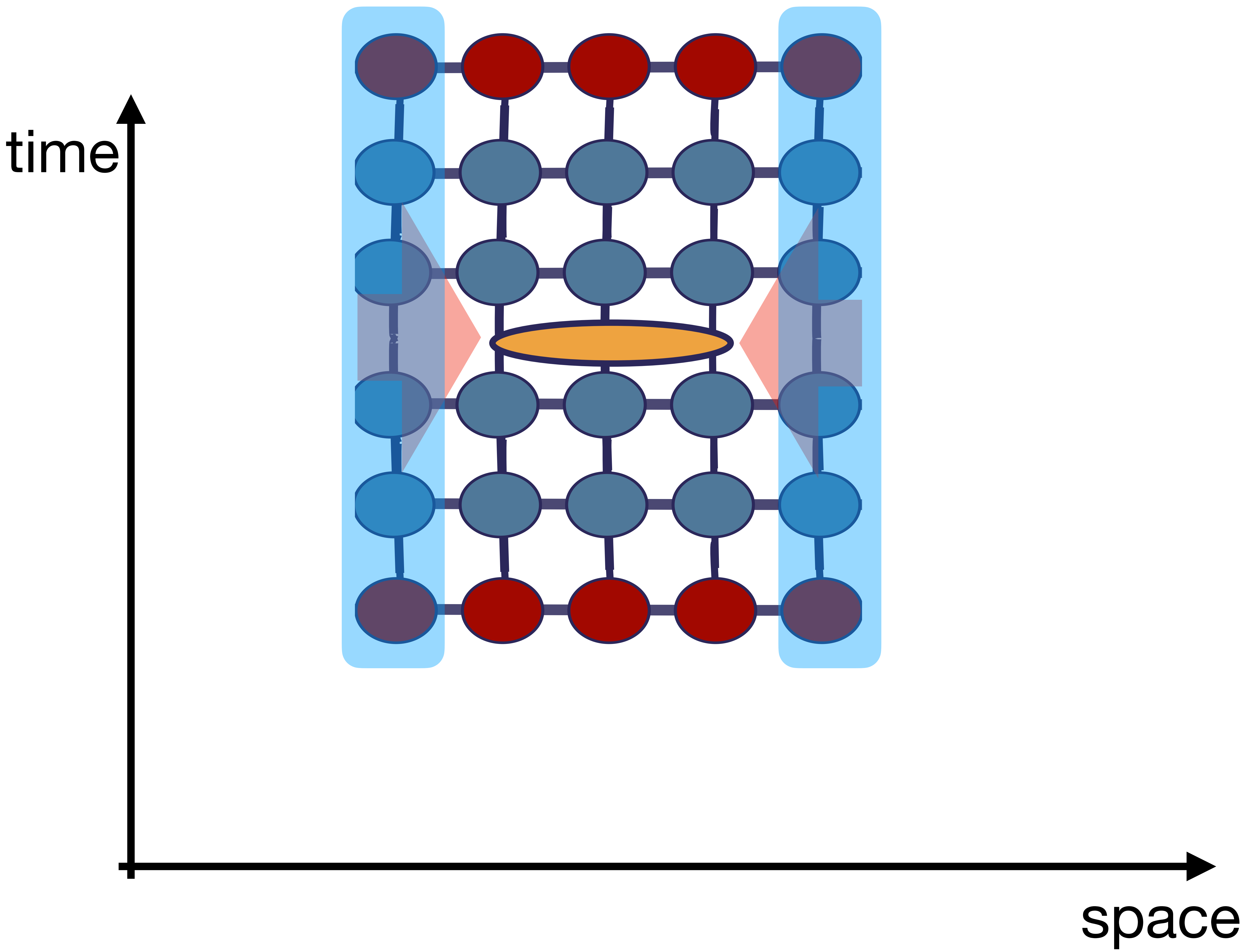


“Transverse” contraction

Transverse contraction



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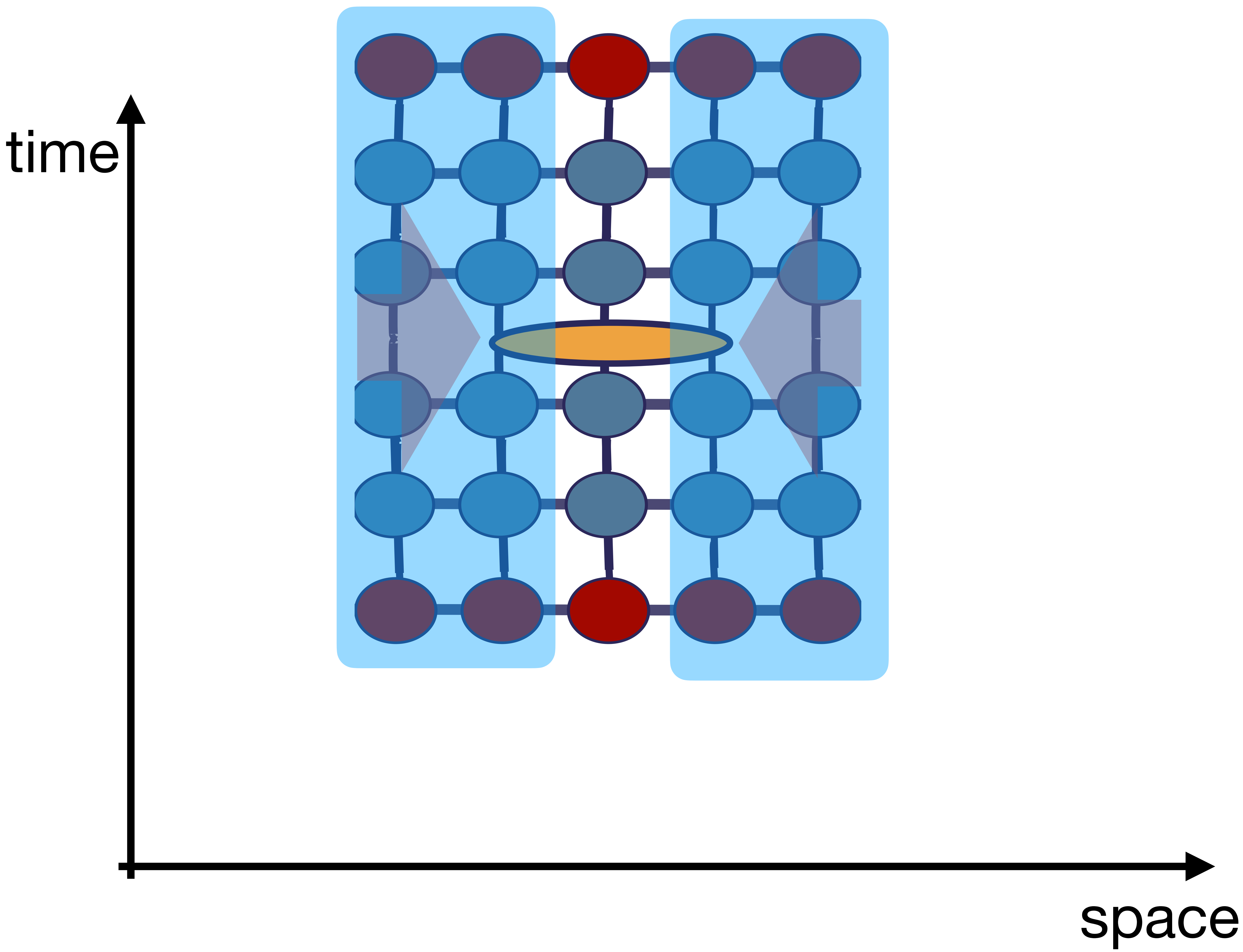
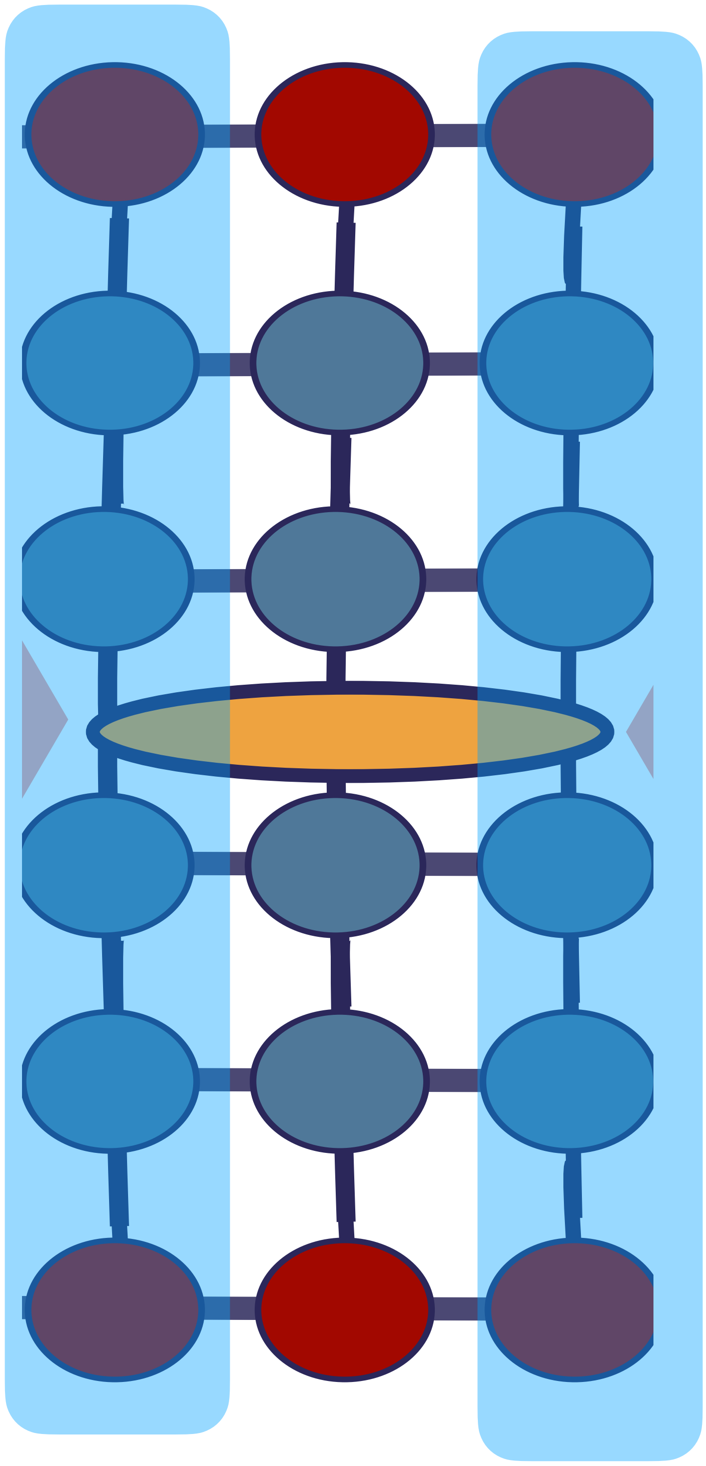


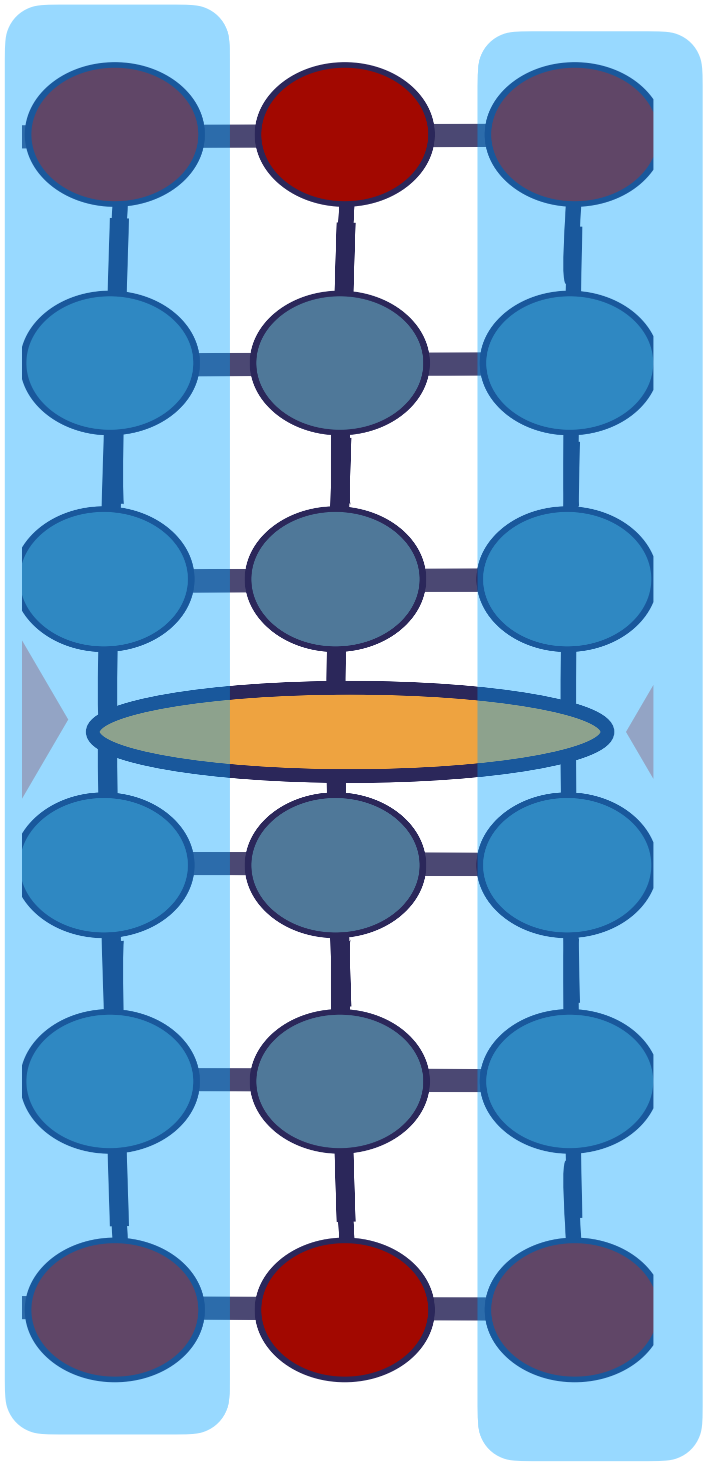
Transverse contraction



Working with TNs makes it easier* to think about alternative contraction paths

*Although if you're smart,
you can probably figure it out anyway
(Feynman and Vernon, Influence functionals ~1963)

Transverse contraction



Are these temporal MPS “compressible” (low entanglement) ?

Can we make some prediction ?

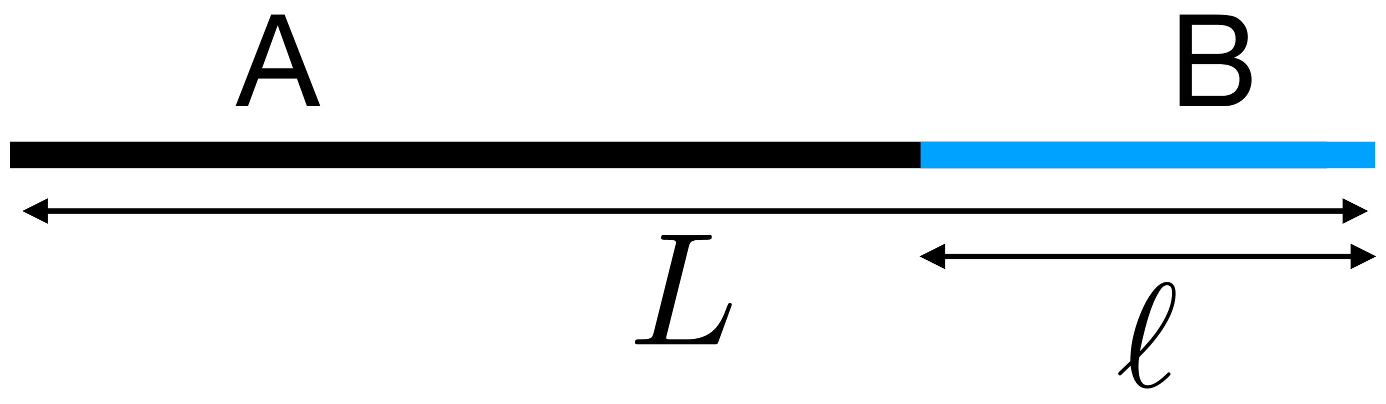
CFT and predictions

Correspondence: for critical models, low-energy/long-range excitations are well described by conformal field theory

CFT predictions: transfer matrices spectra,
effects of boundary conditions, entropies…

“famous” results: entanglement entropies for ground state of chain (Calabrese and Cardy, 2004)

$$S \sim c \log\left[\frac{L}{\pi}\sin\left(\frac{\pi\ell}{L}\right)\right]$$



$$S \sim c\log(\ell)$$

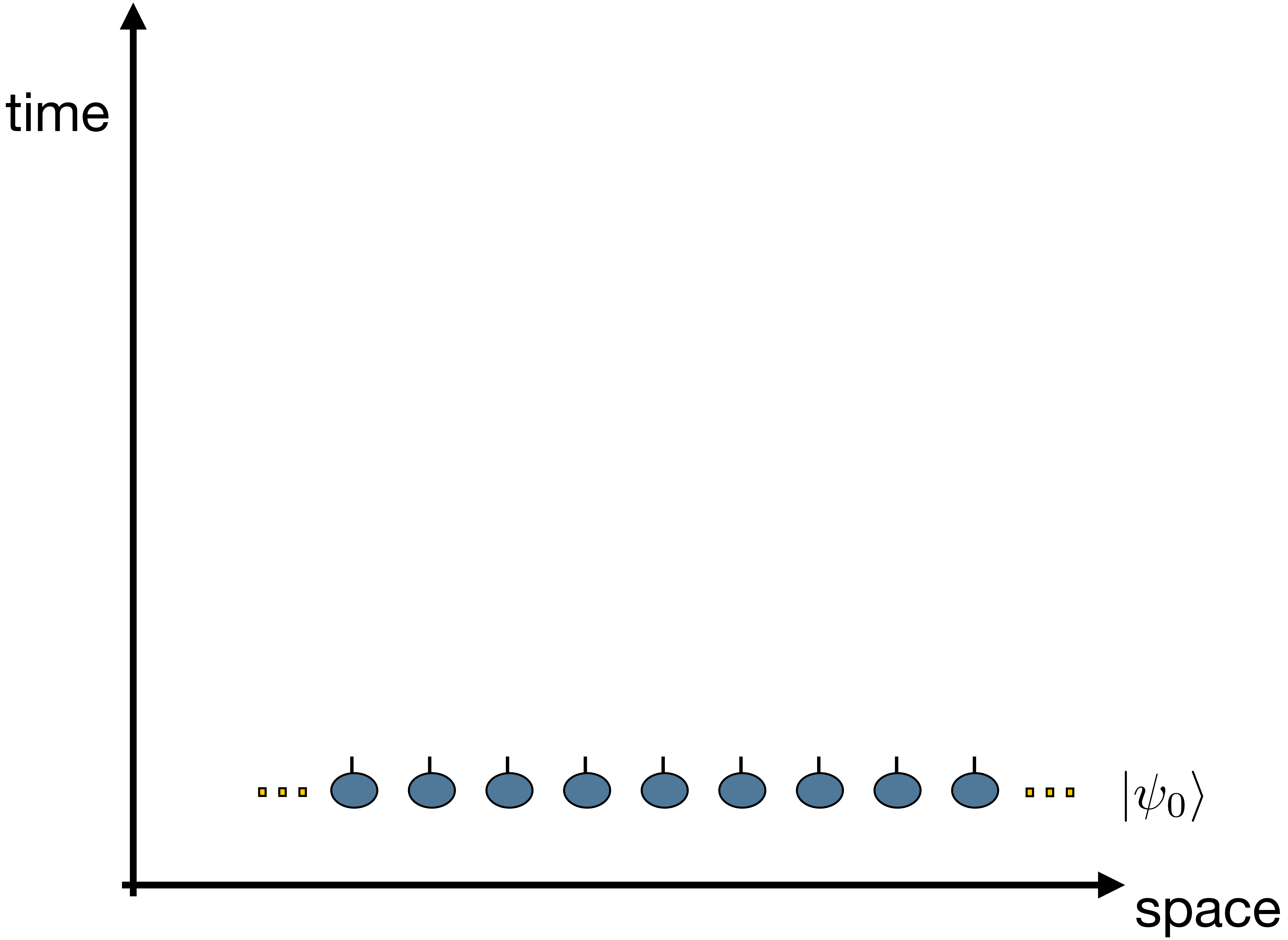
Mapping CFT: Loschmidt echo

Our setup: Global quench from product state of infinite system
with critical Hamiltonian

$$|\psi_0\rangle \qquad \longrightarrow \qquad |\psi(t)\rangle = e^{-iH_ct}\,|\psi_0\rangle$$

Return probability: “Loschmidt echo” $$L = \langle\psi_0|e^{-iH_ct}|\psi_0\rangle$$

Mapping CFT: Loschmidt echo



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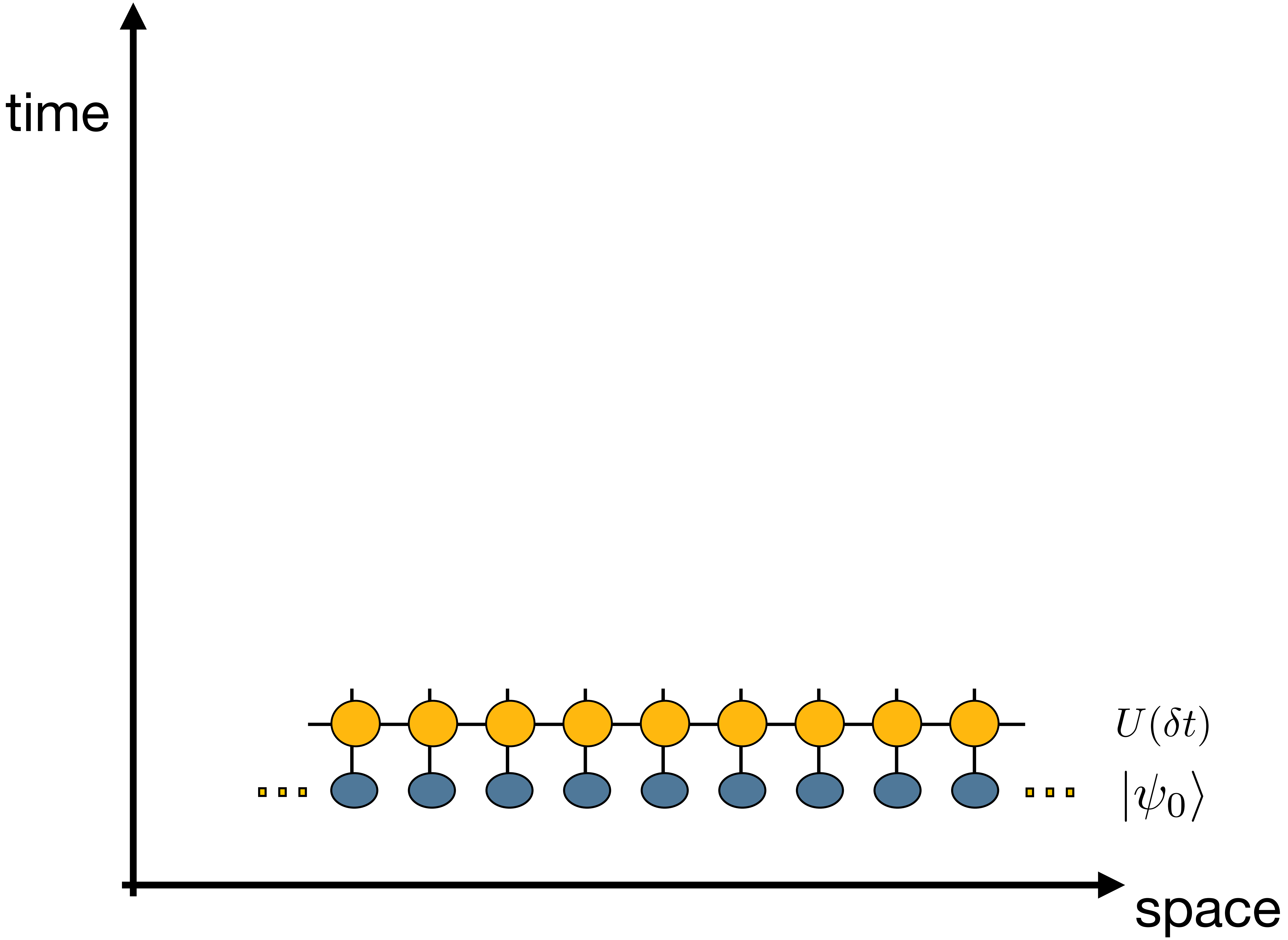
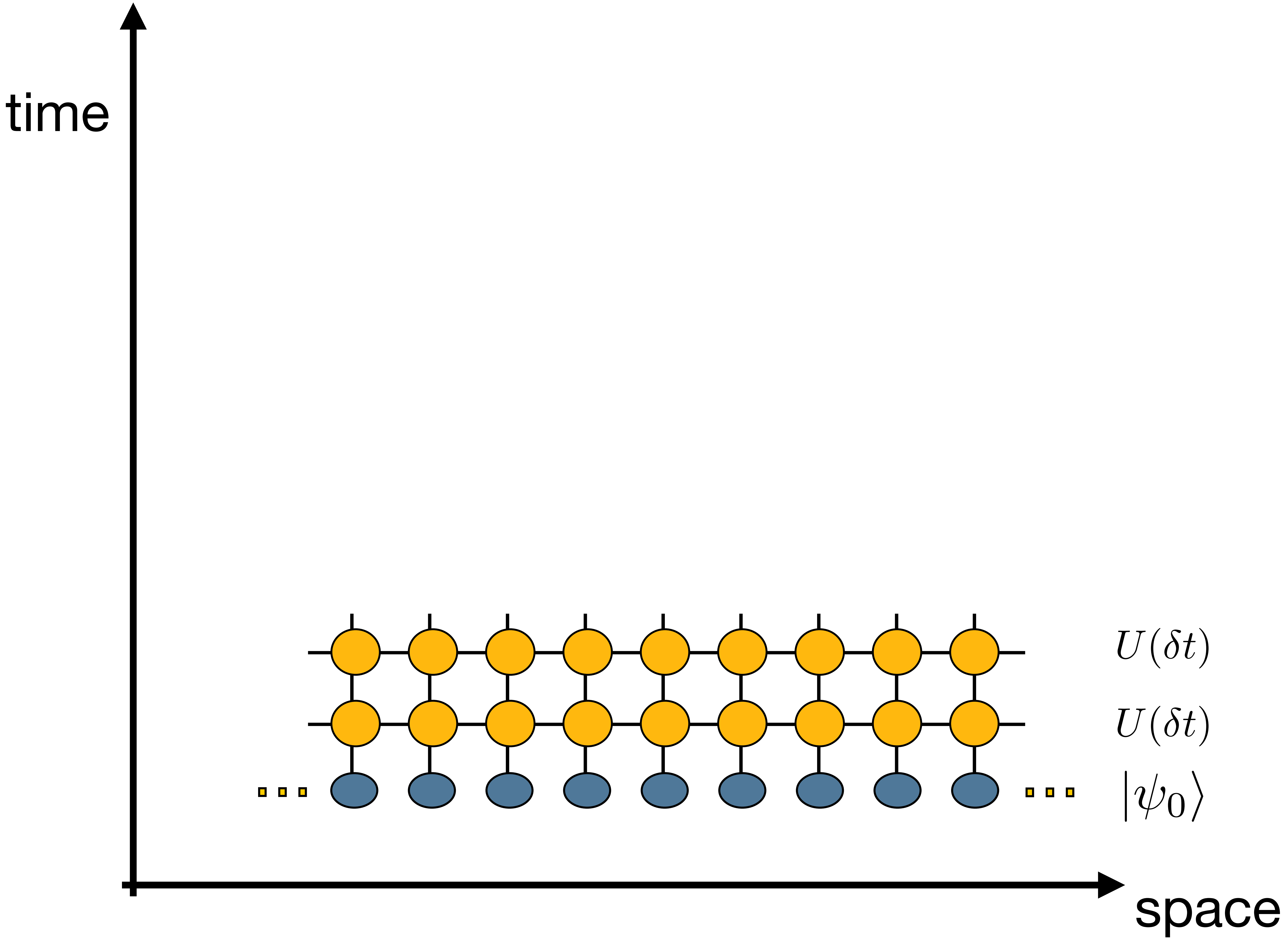


Mapping CFT: Loschmidt echo



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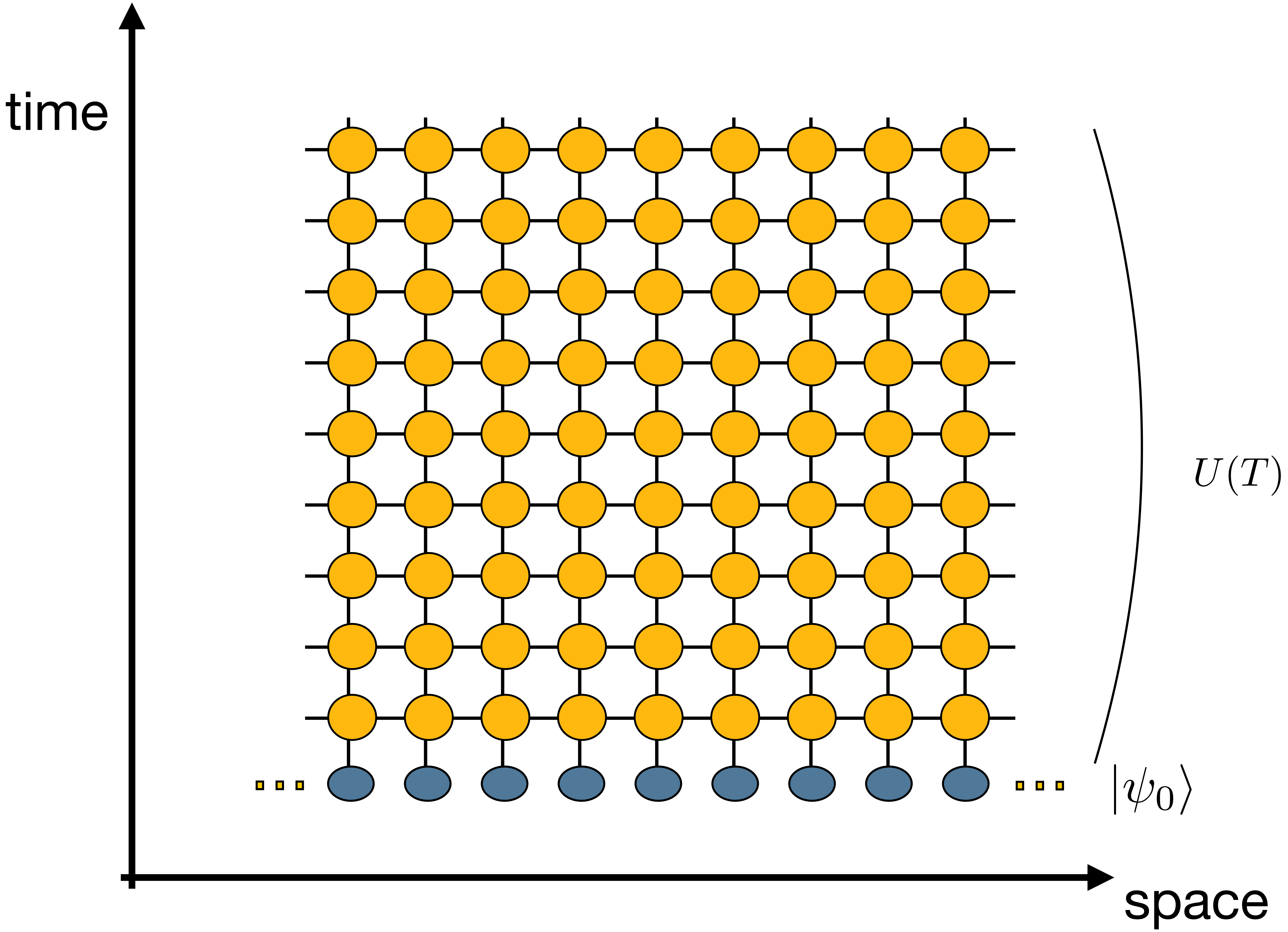

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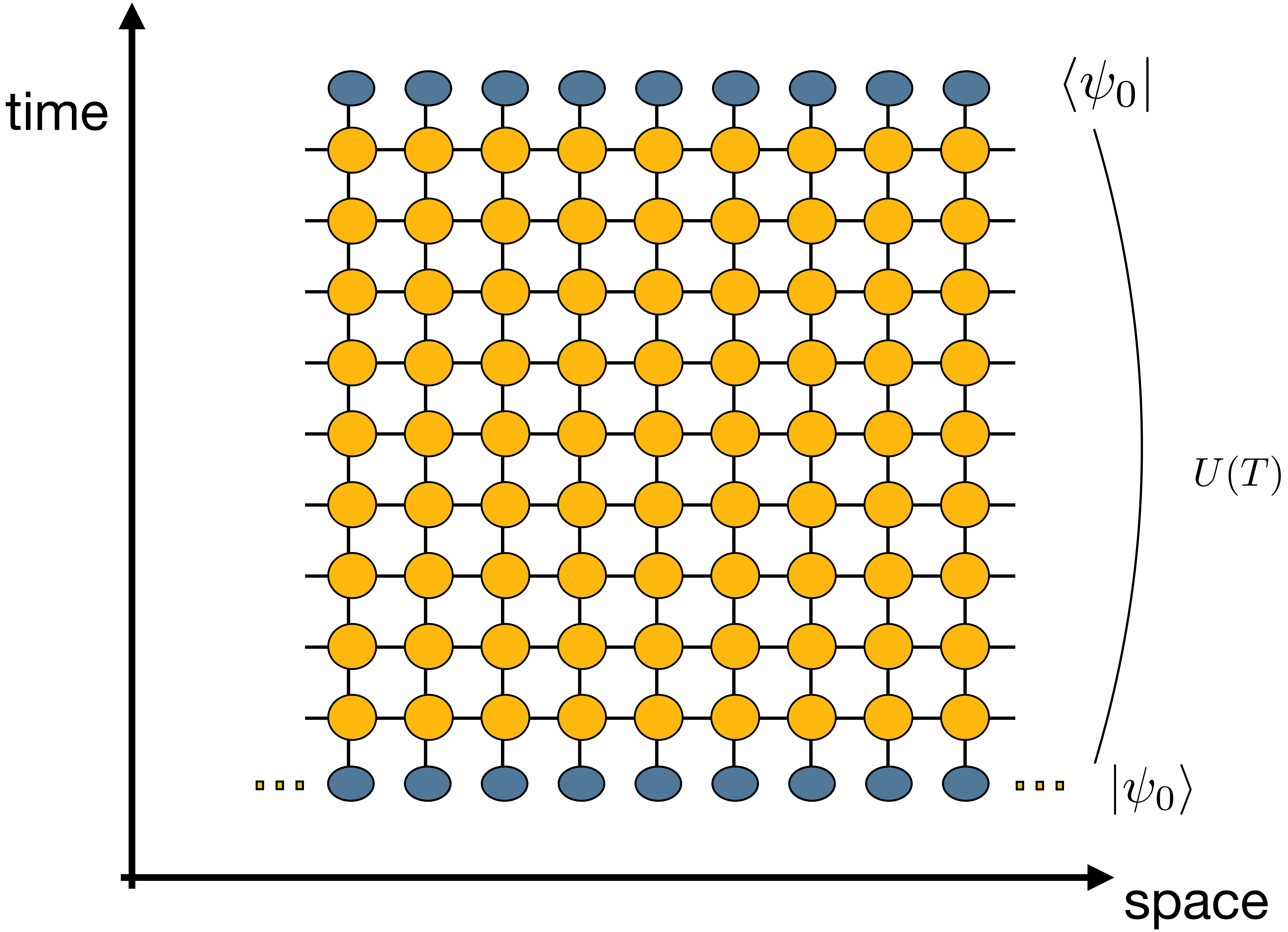

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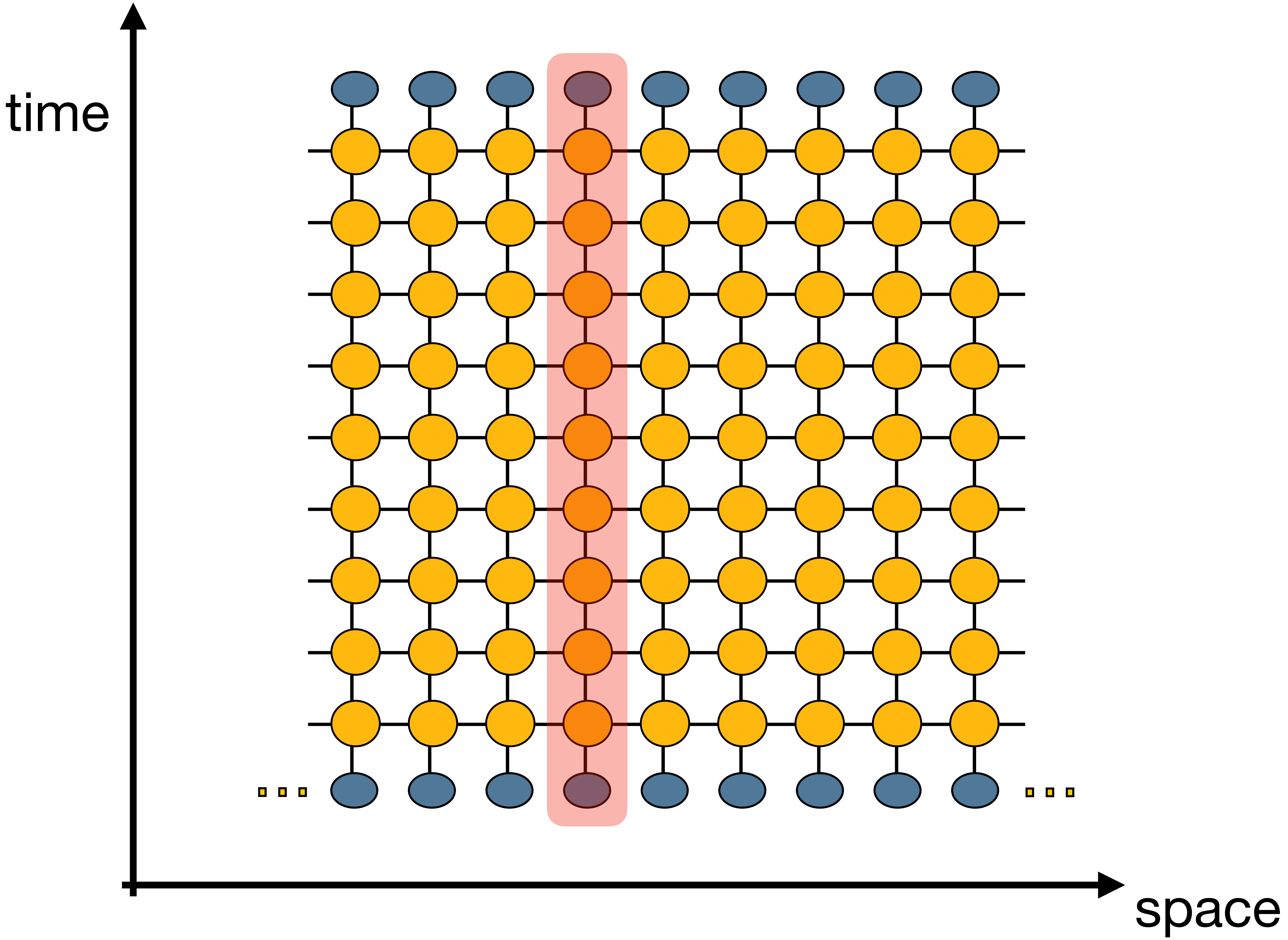

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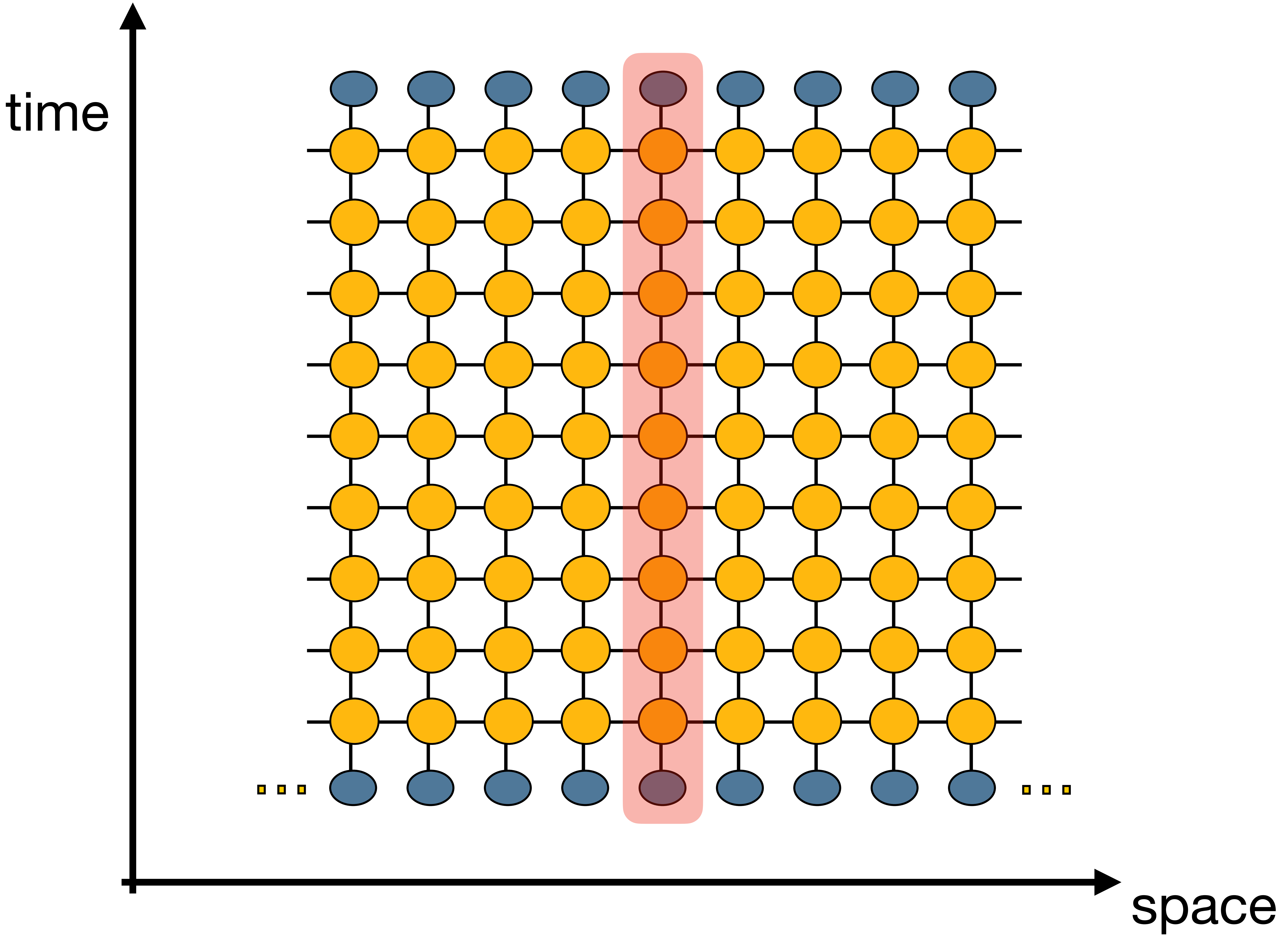

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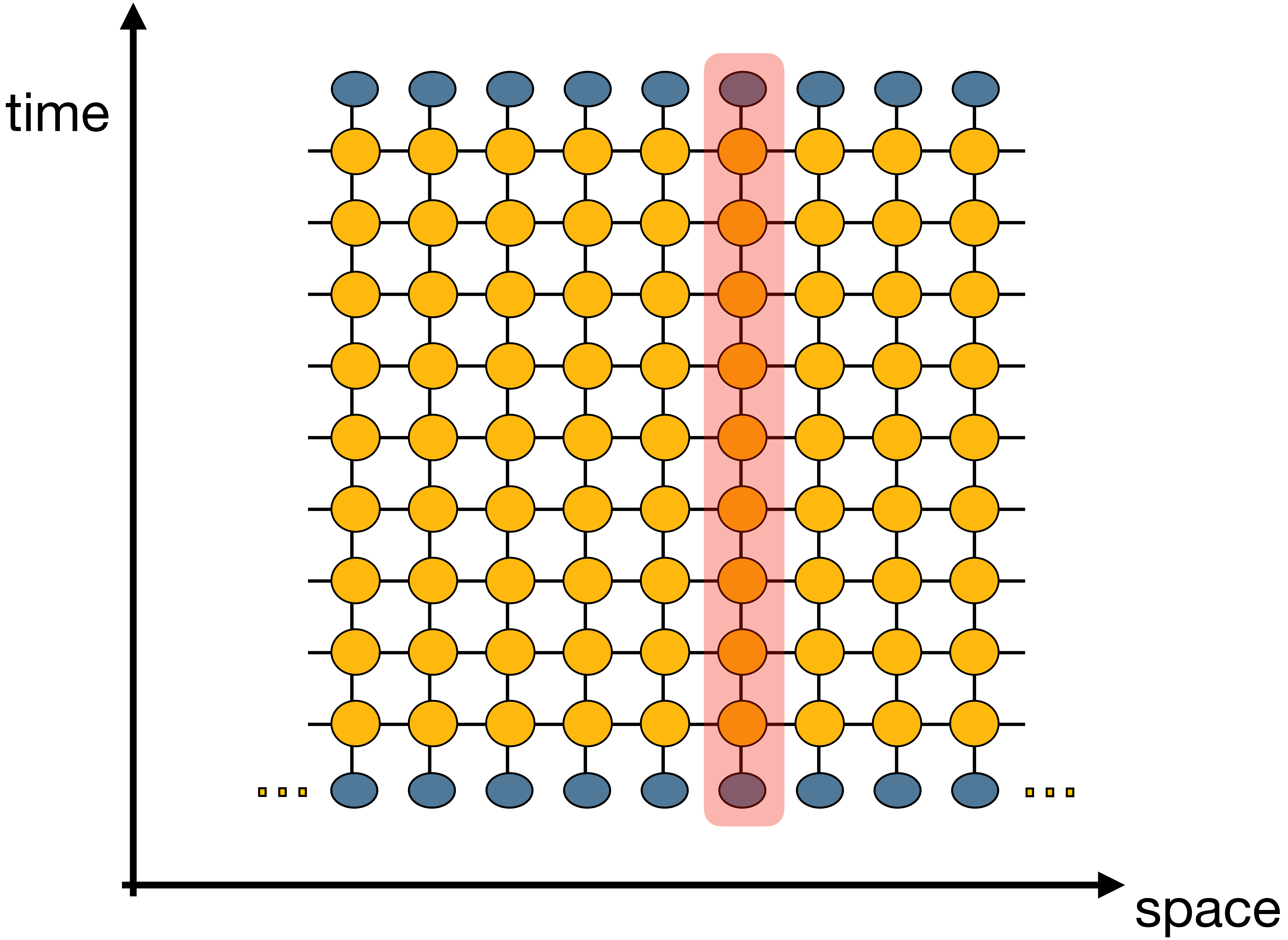

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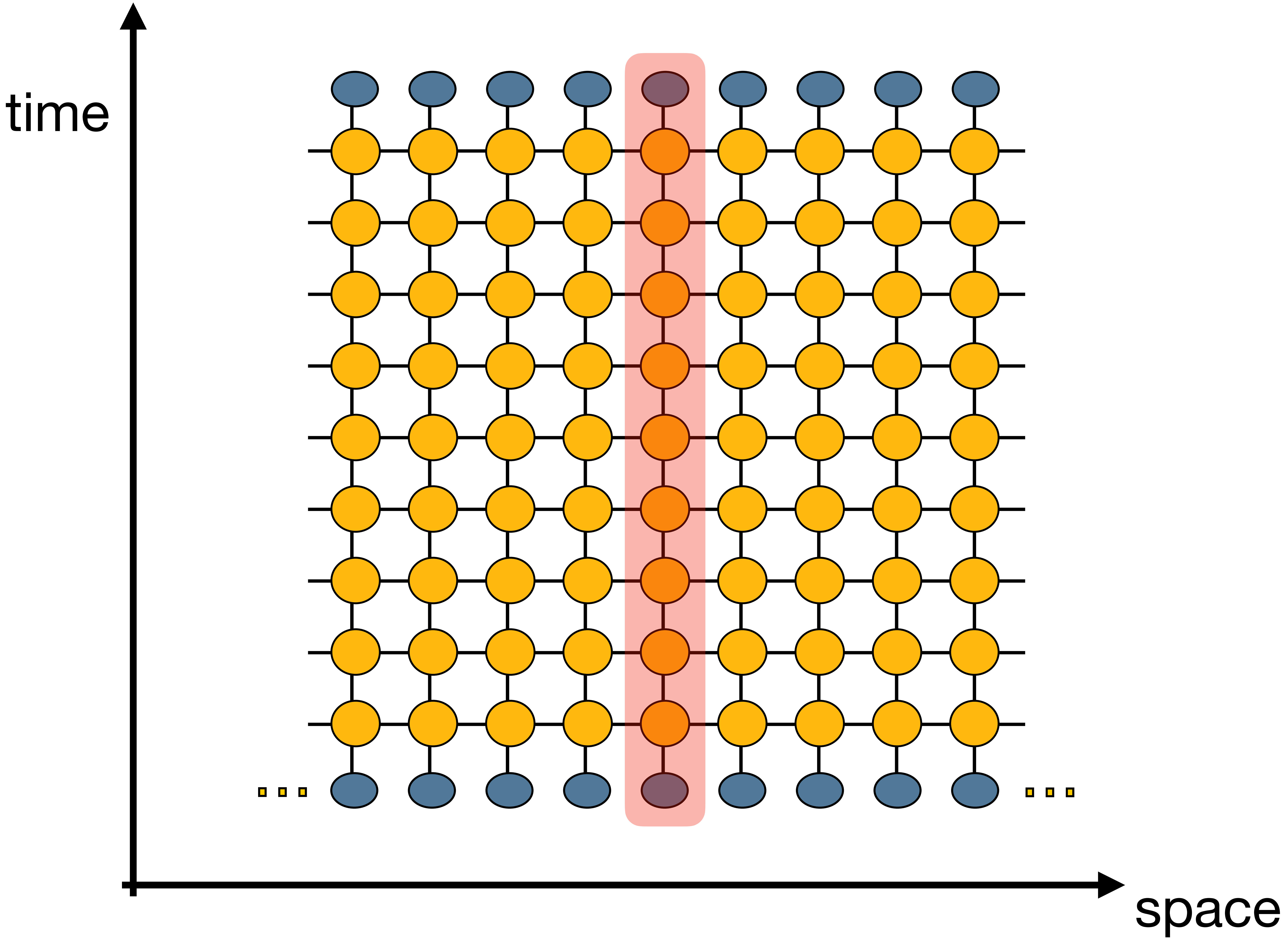

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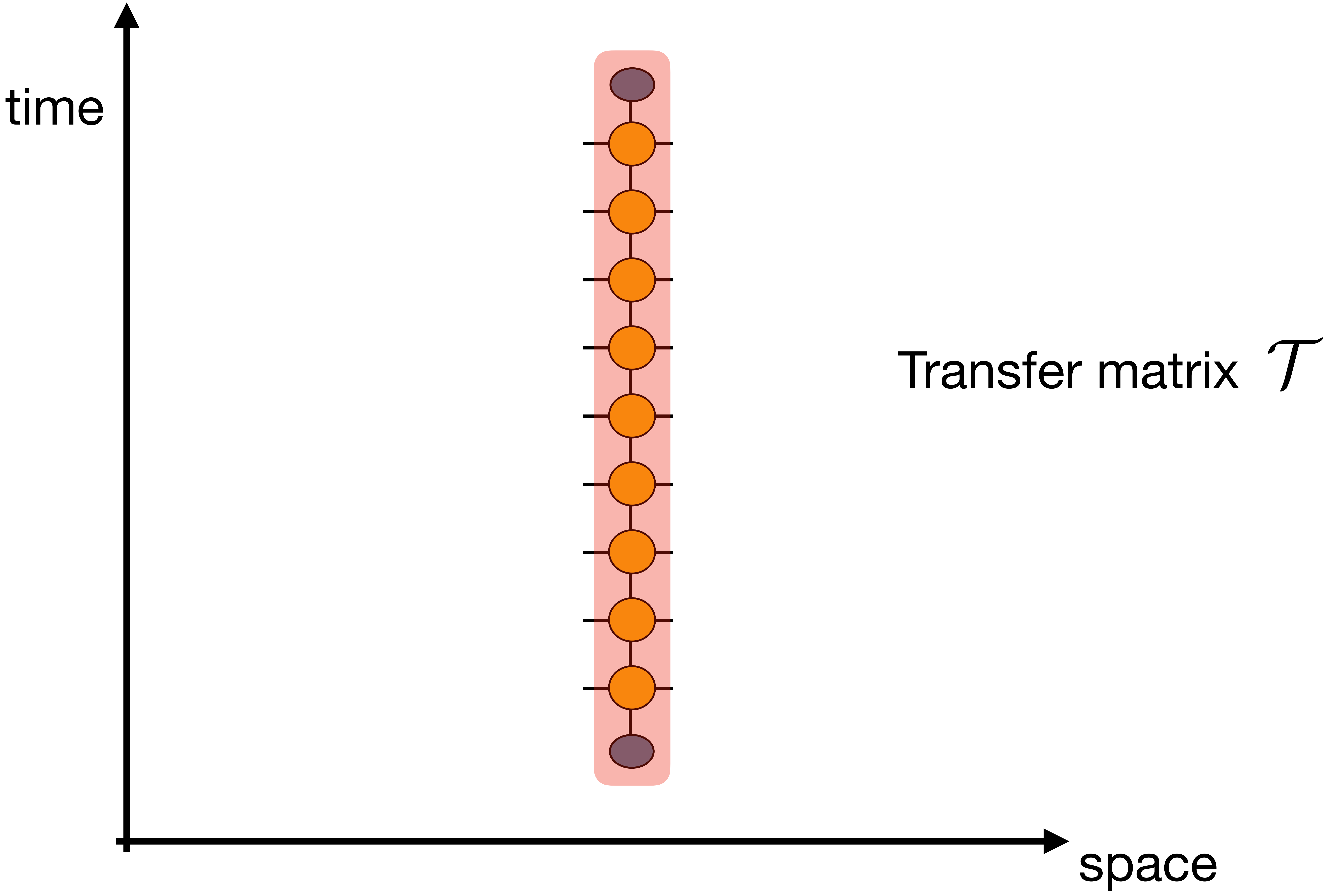
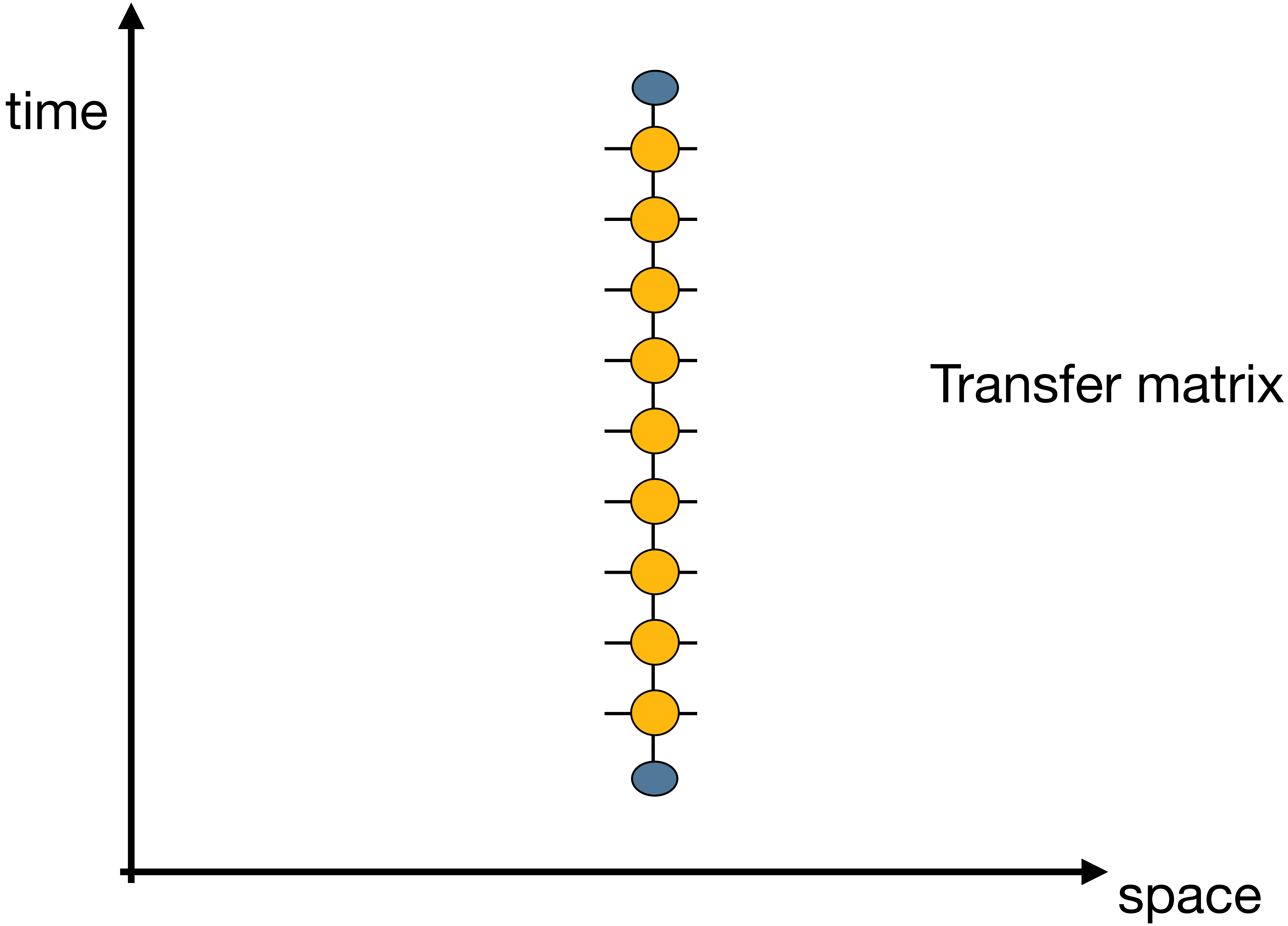
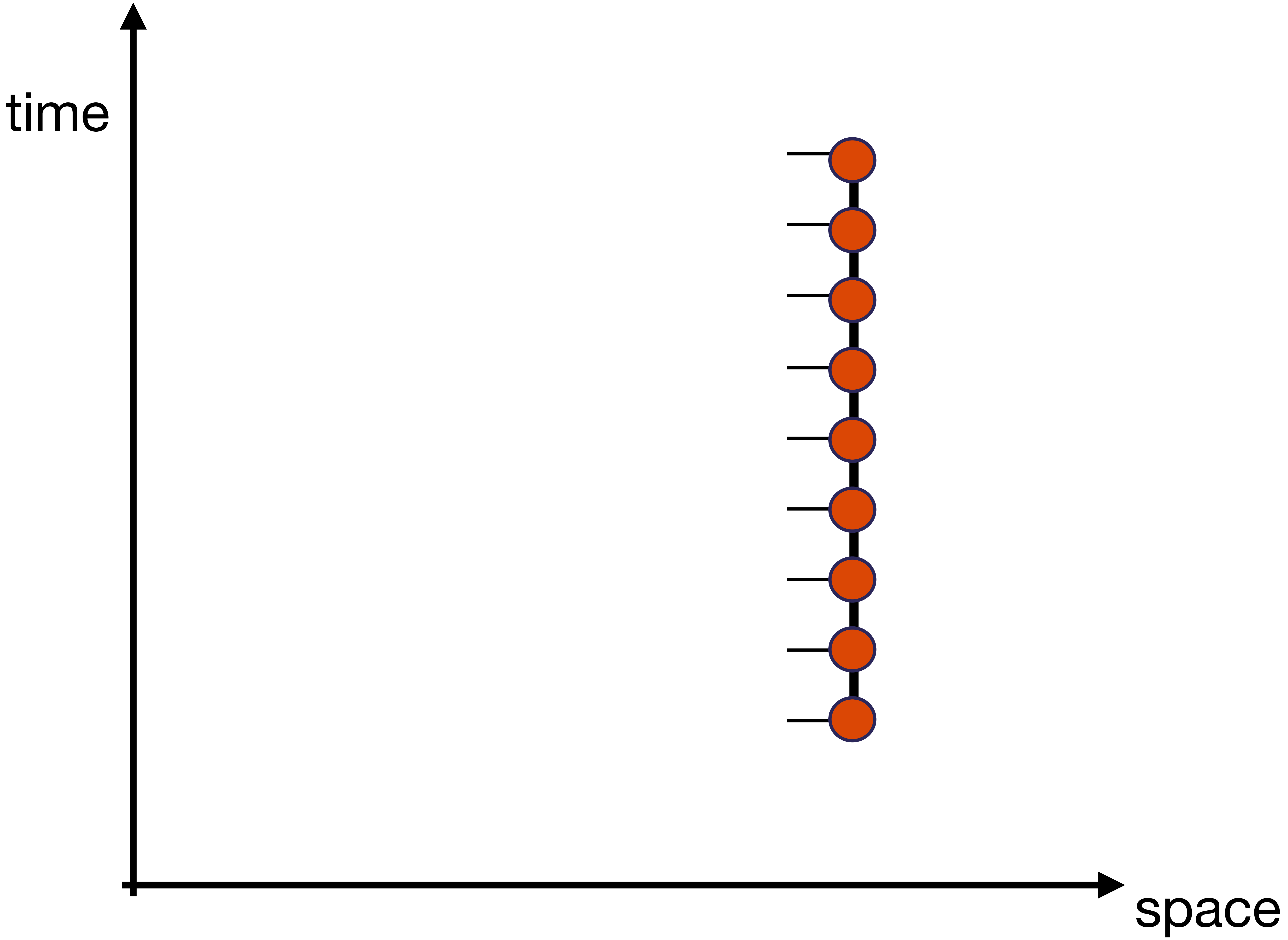


Contracting the network: Dominant vectors



Transfer matrix $\mathcal{T}$

Contracting the network: Dominant vectors



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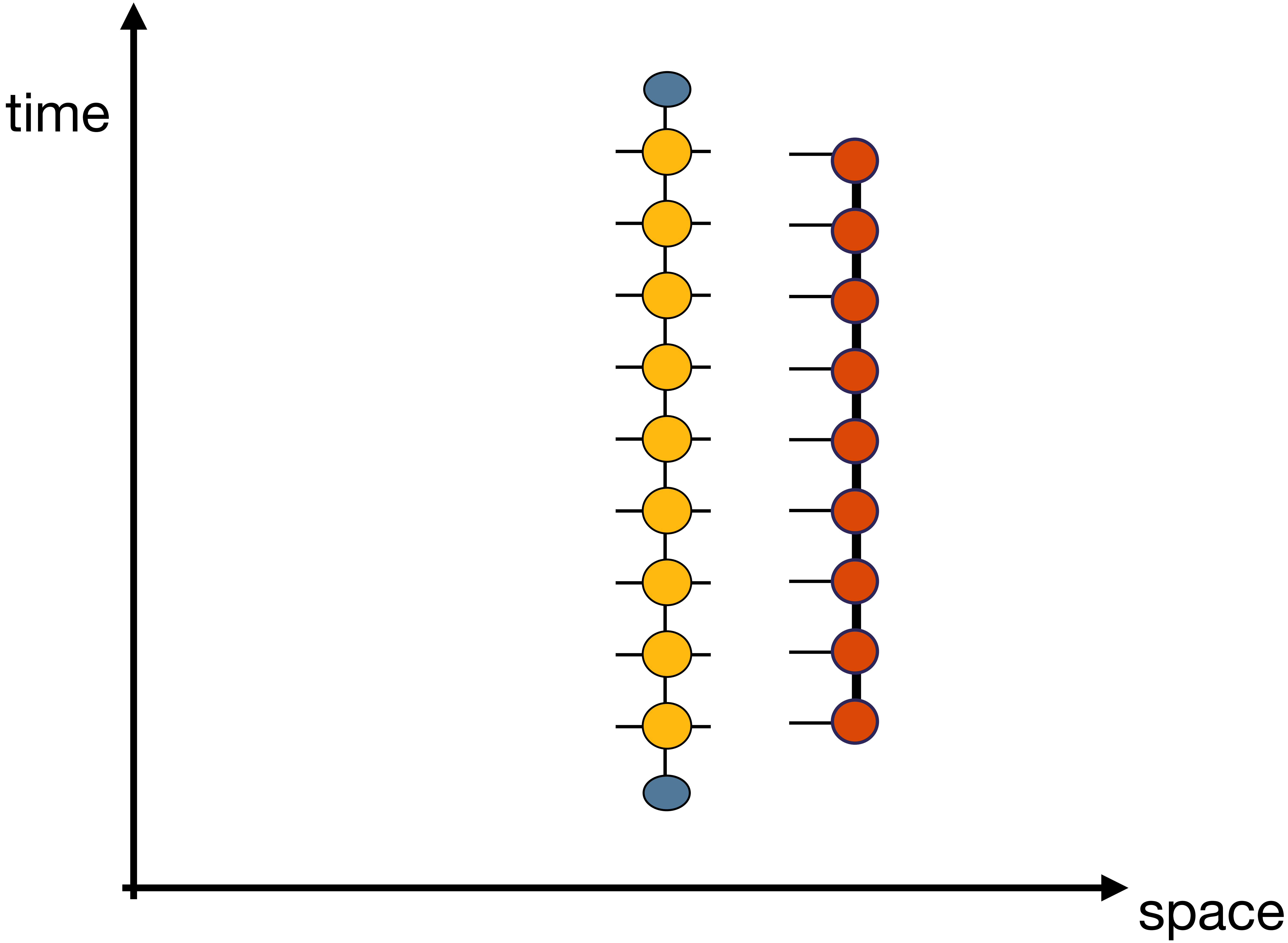

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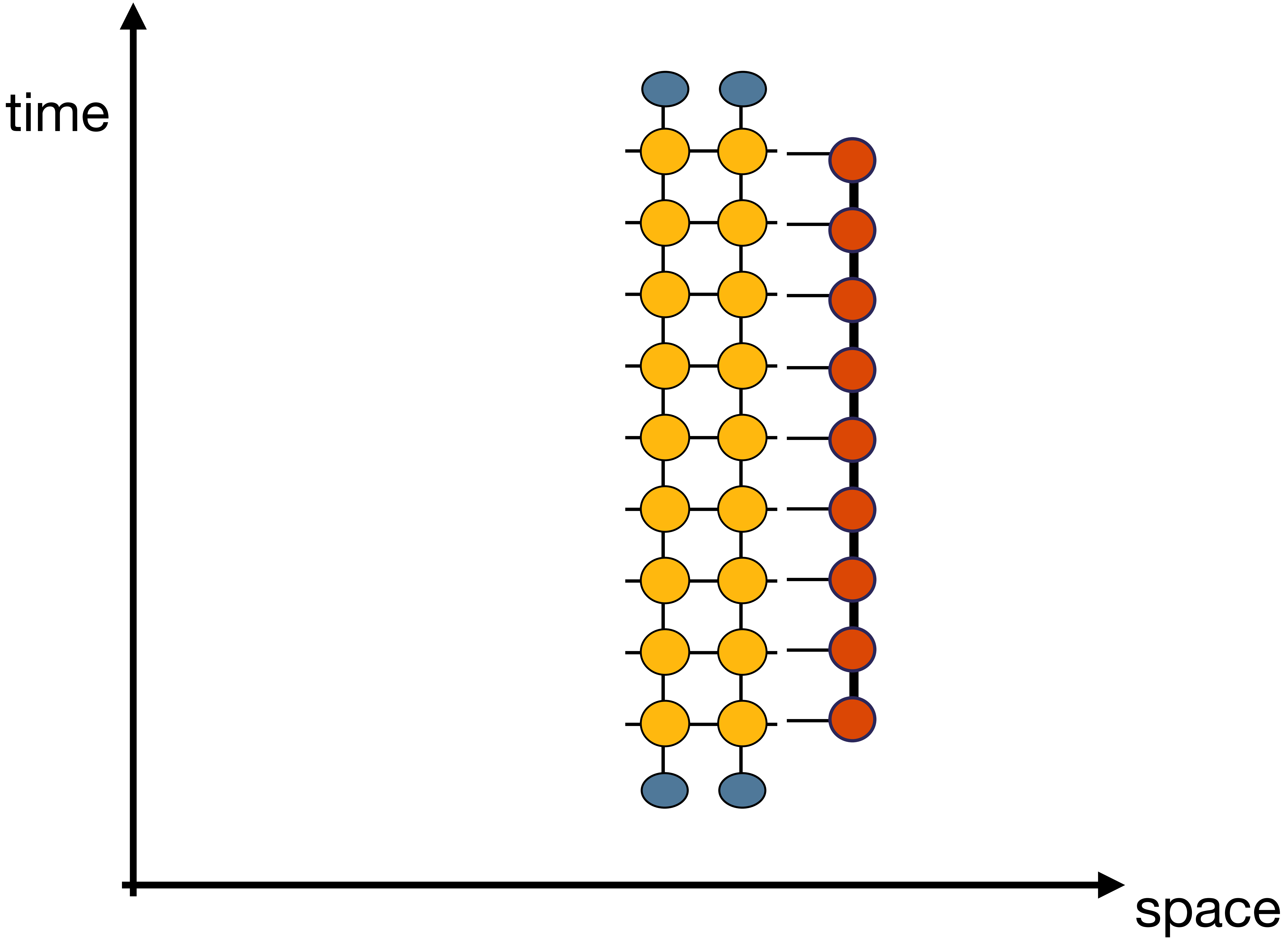

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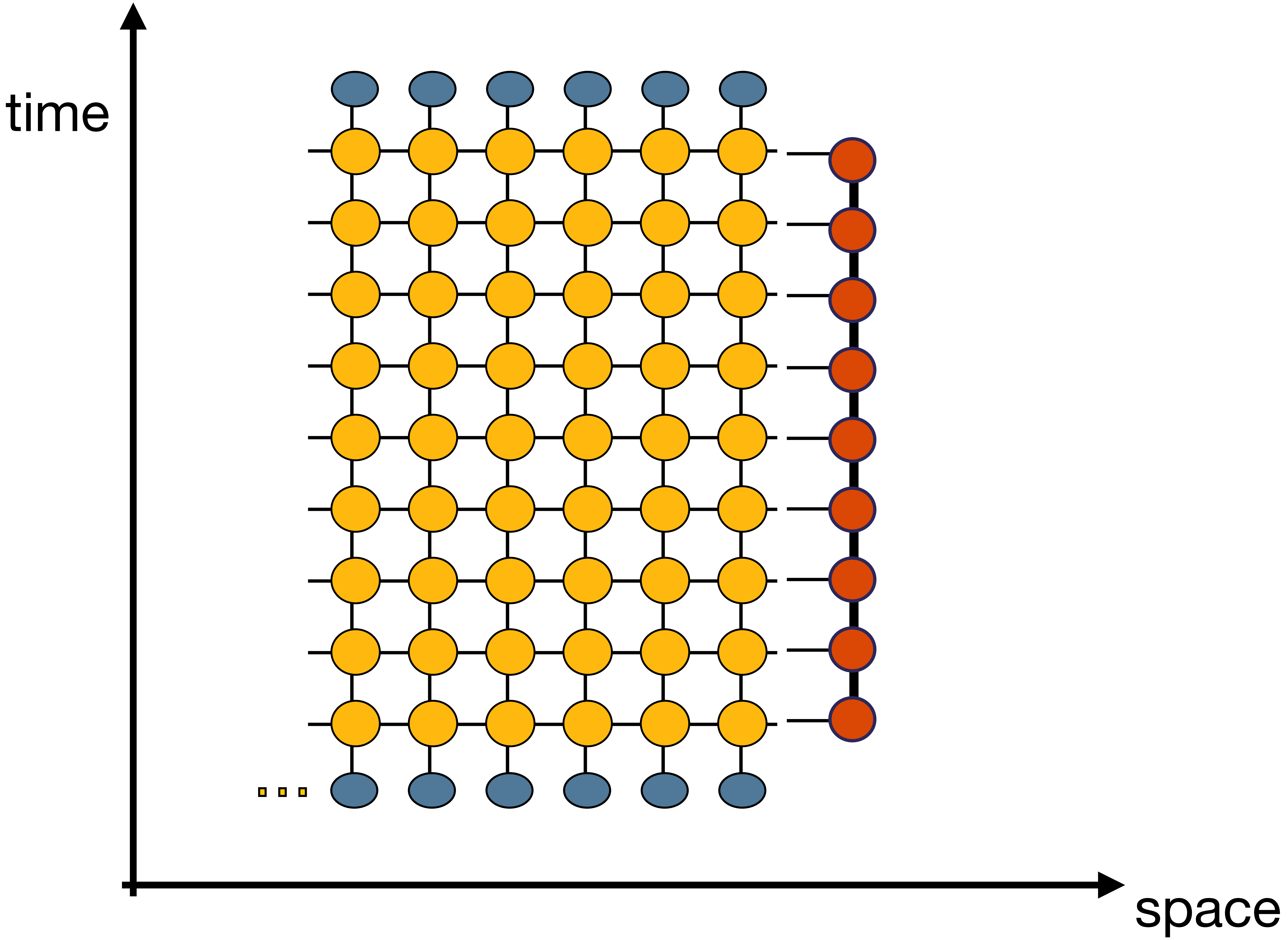

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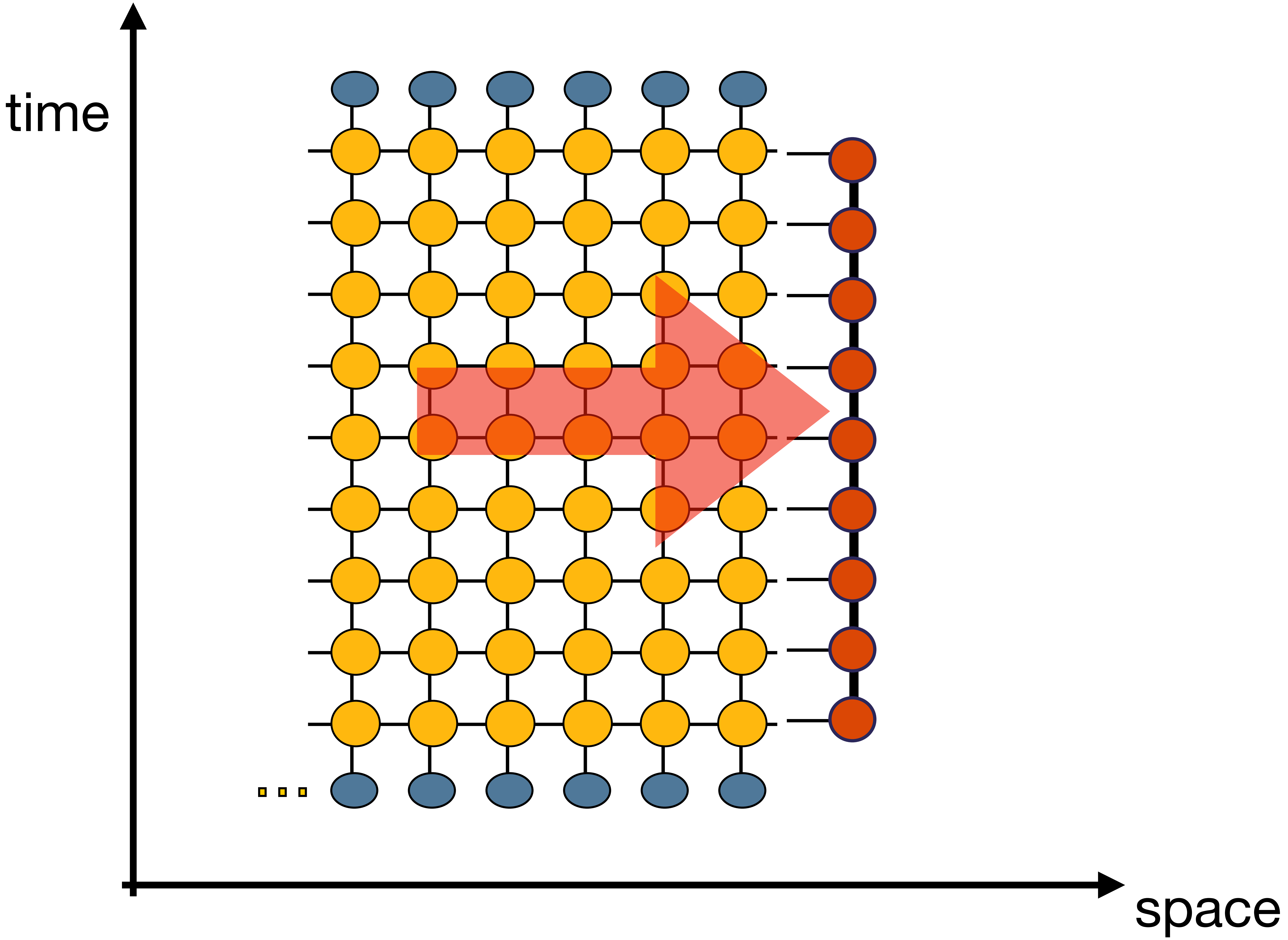

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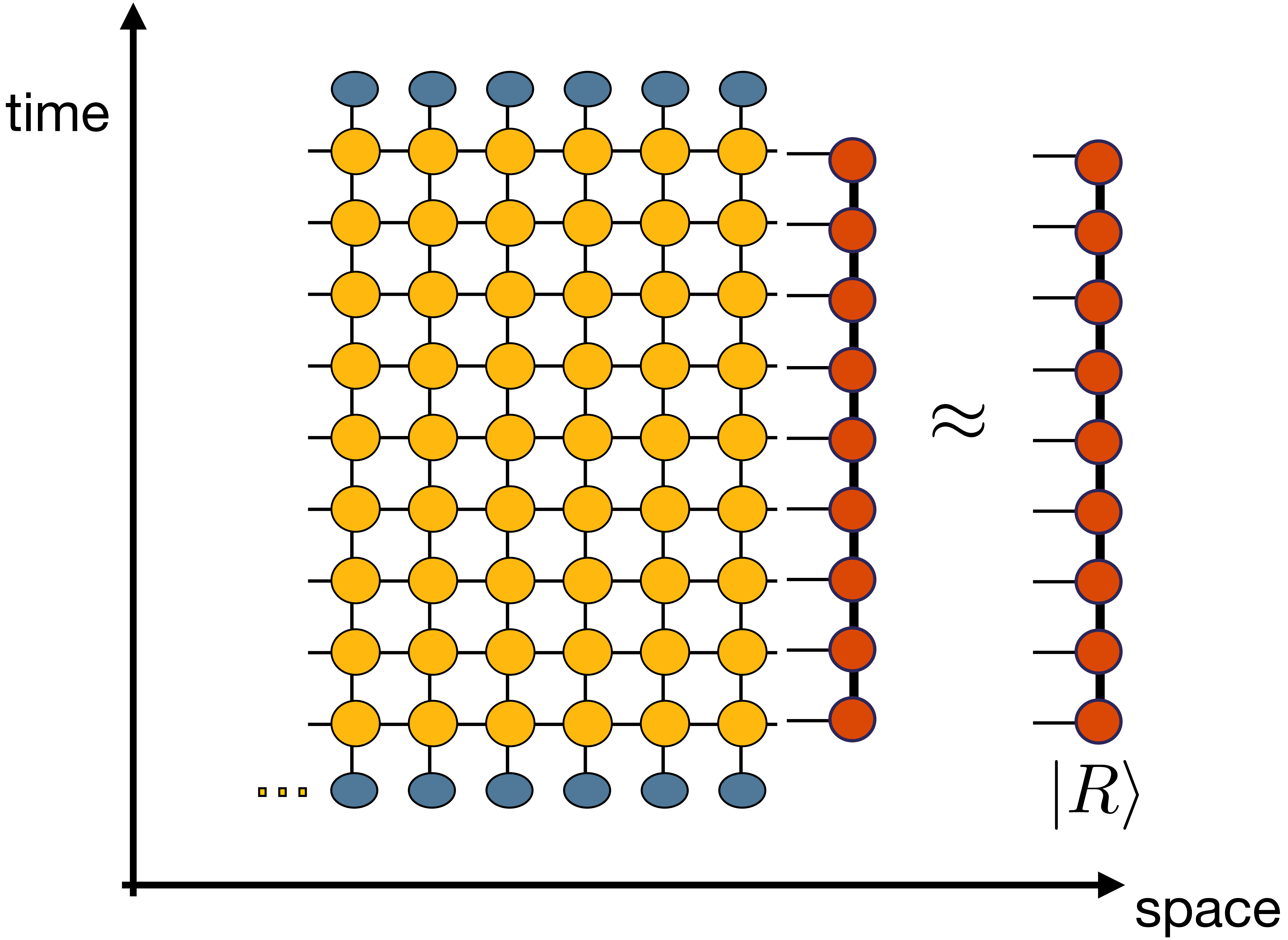

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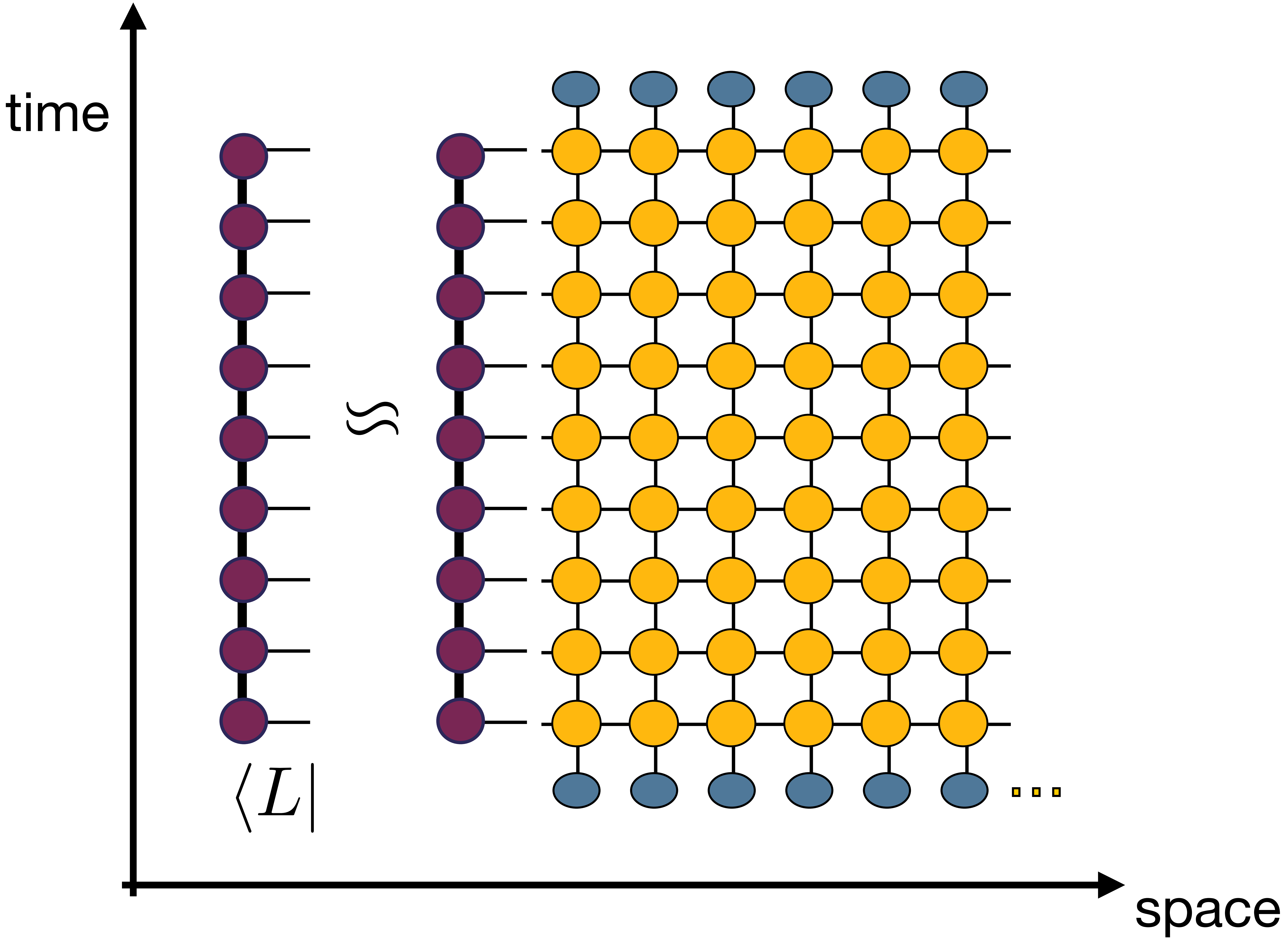

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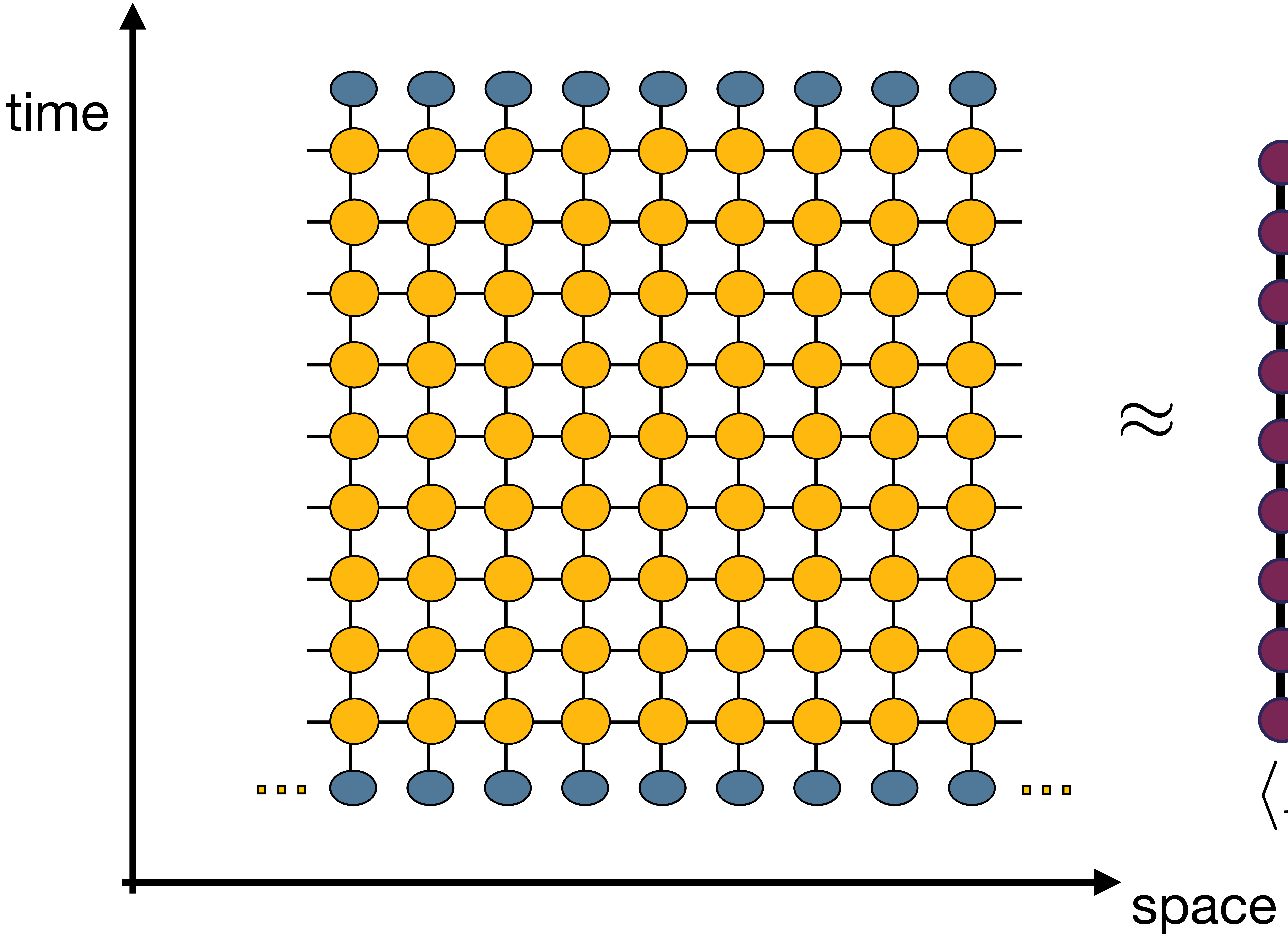

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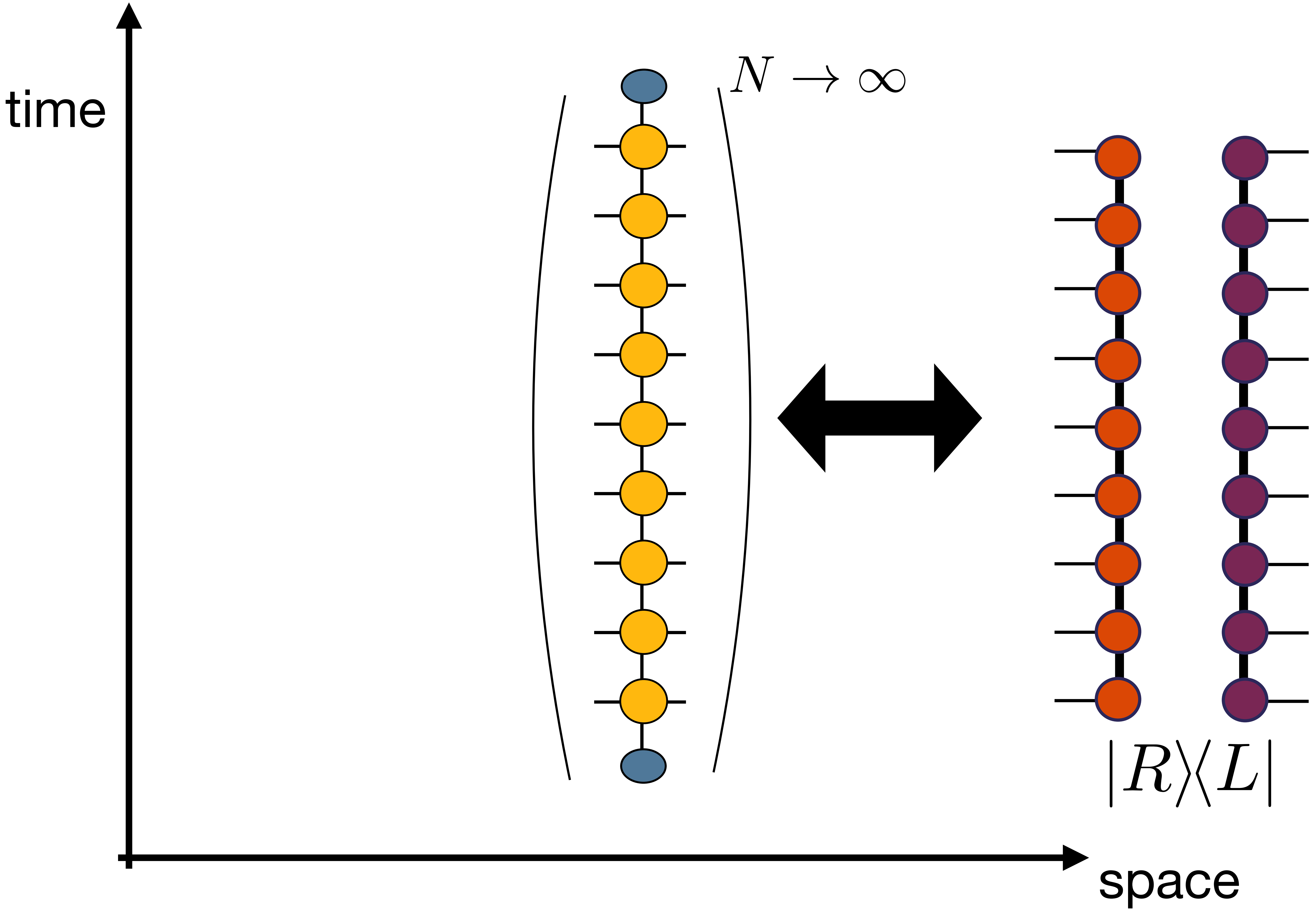
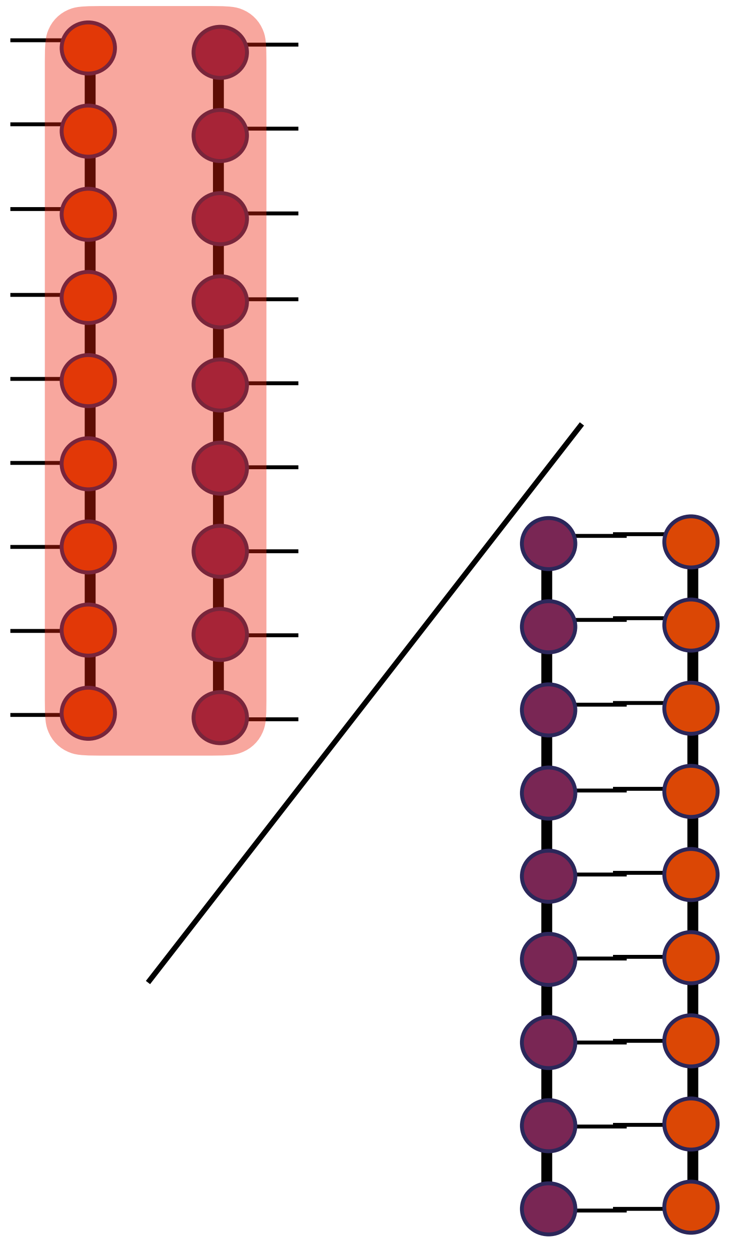


Transition matrix

$$\mathcal{T} = \frac{|R\rangle \langle L|}{\langle L|R\rangle}$$



Recently proposed in AdS/CFT

Y. Nakata, T. Takayanagi, Y. Taki, K. Tamaoka, and Z. Wei, PRD (2021) 103, 026005

K. Doi, et al, arxiv:2302.11695

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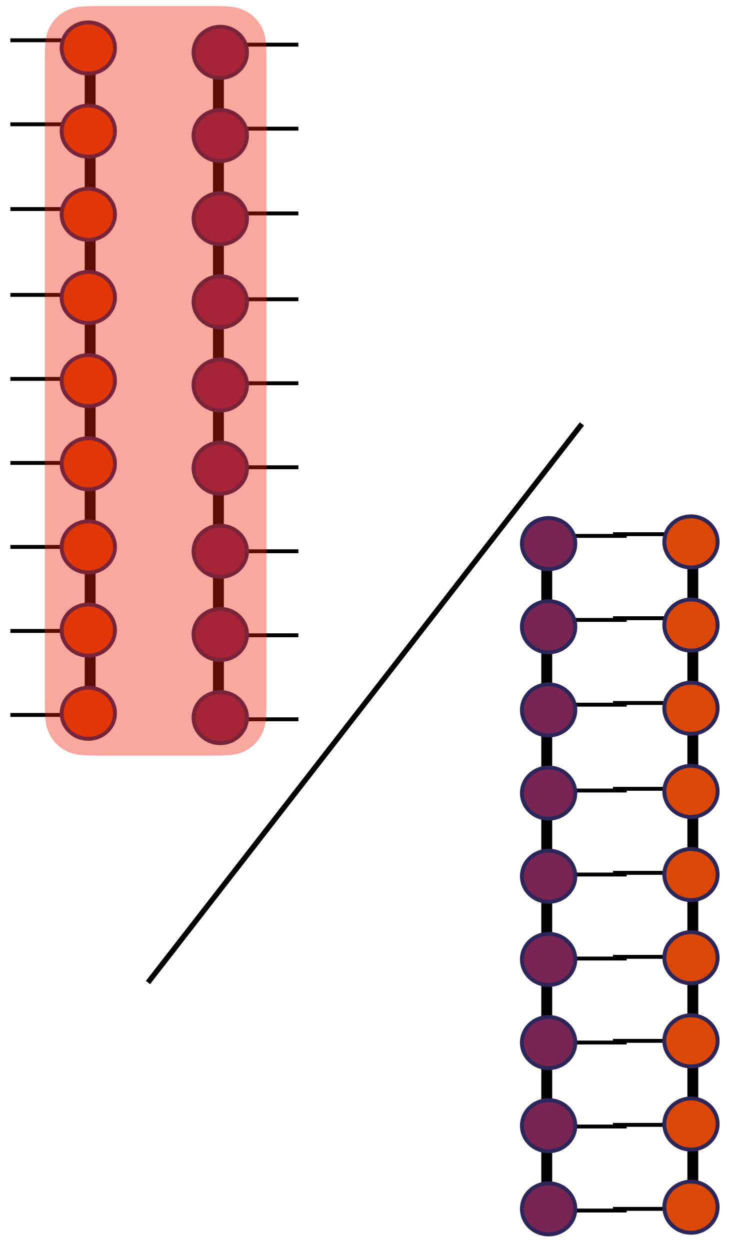

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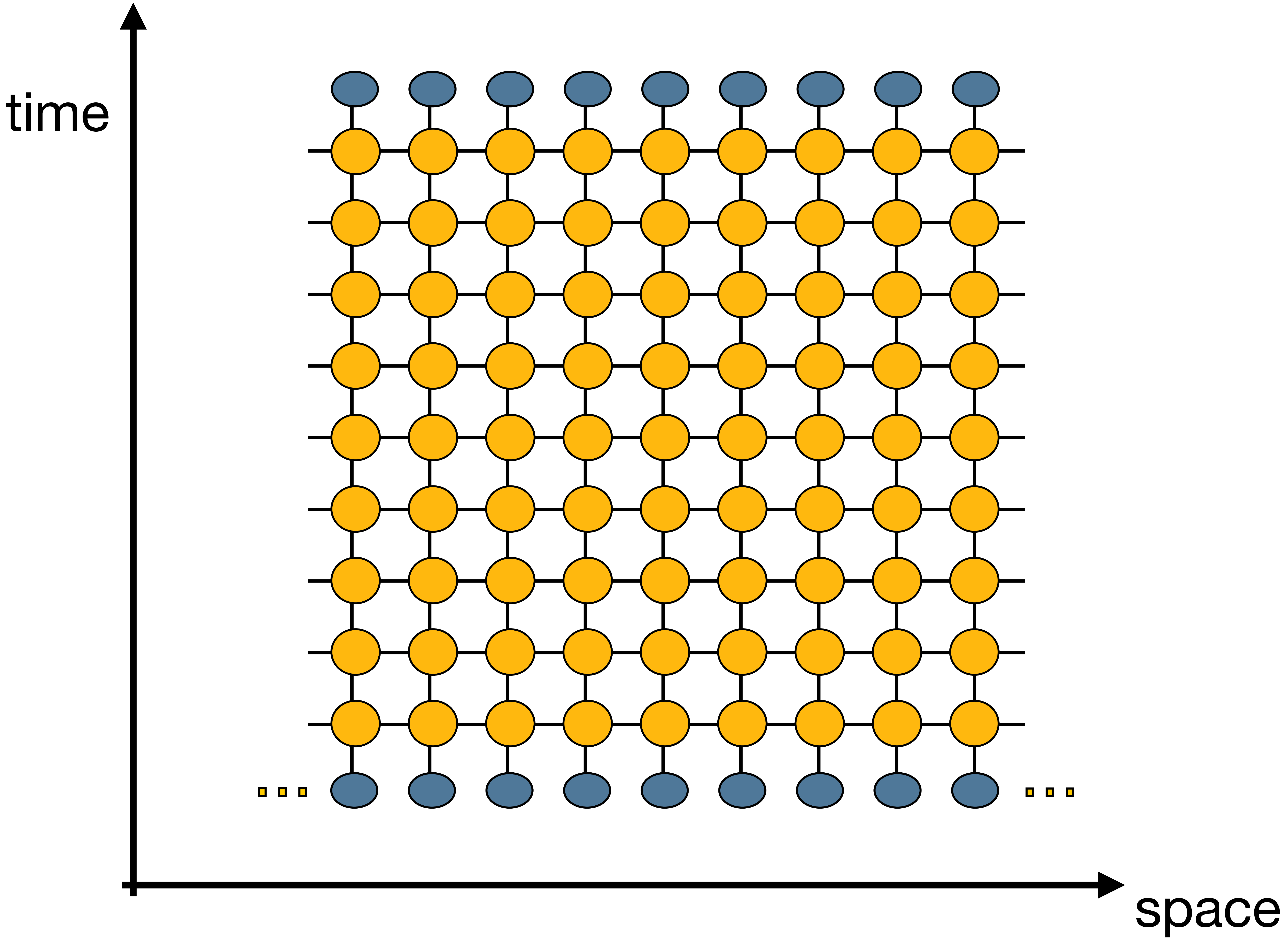
From its eigenvalues: Generalized temporal entropy

Related to computational complexity of contracting TN

SC, C.Ramos and L.Tagliacozzo, PRR 6 (2024) 3, 033021

Mapping CFT: Loschmidt echo



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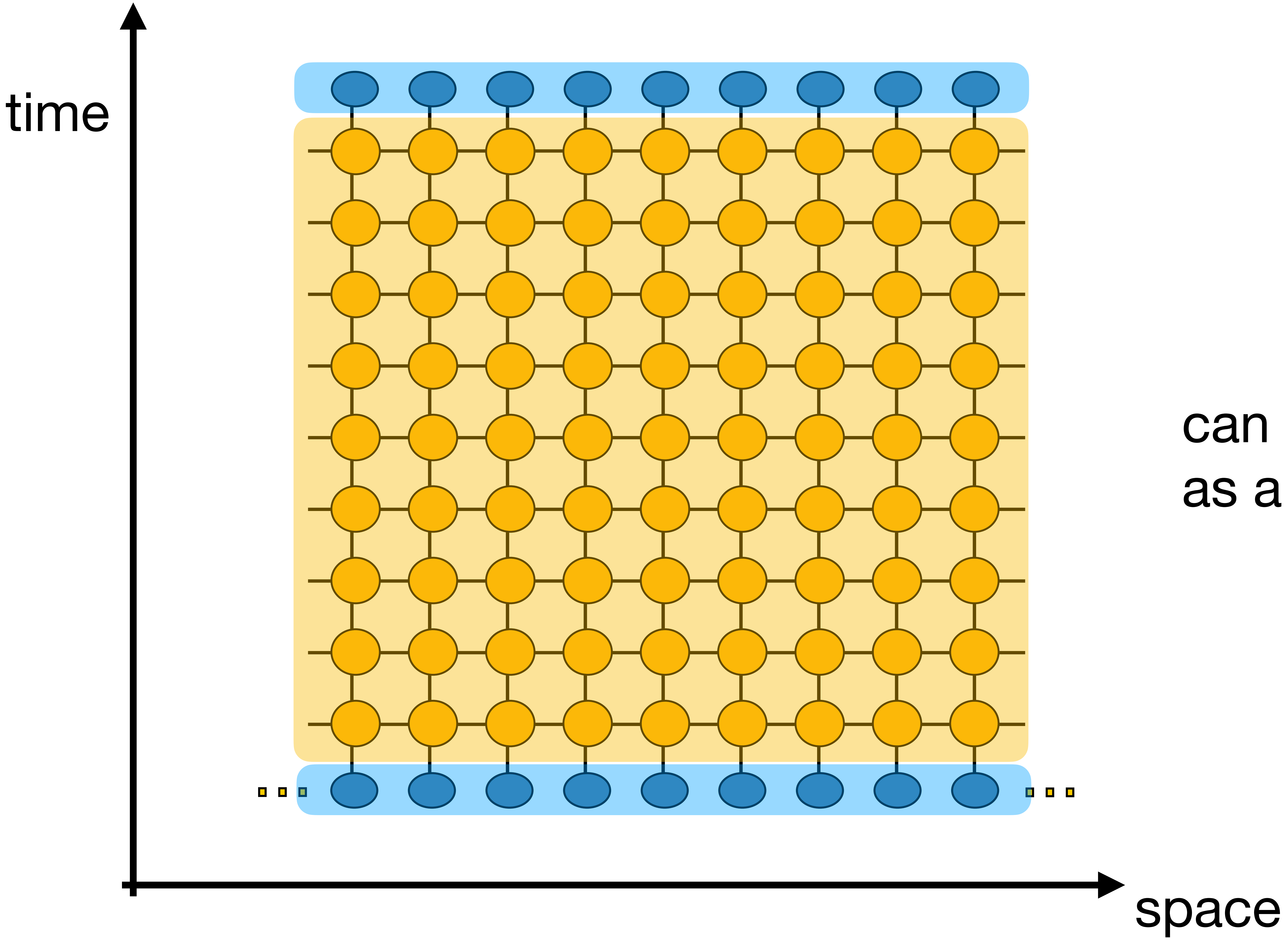
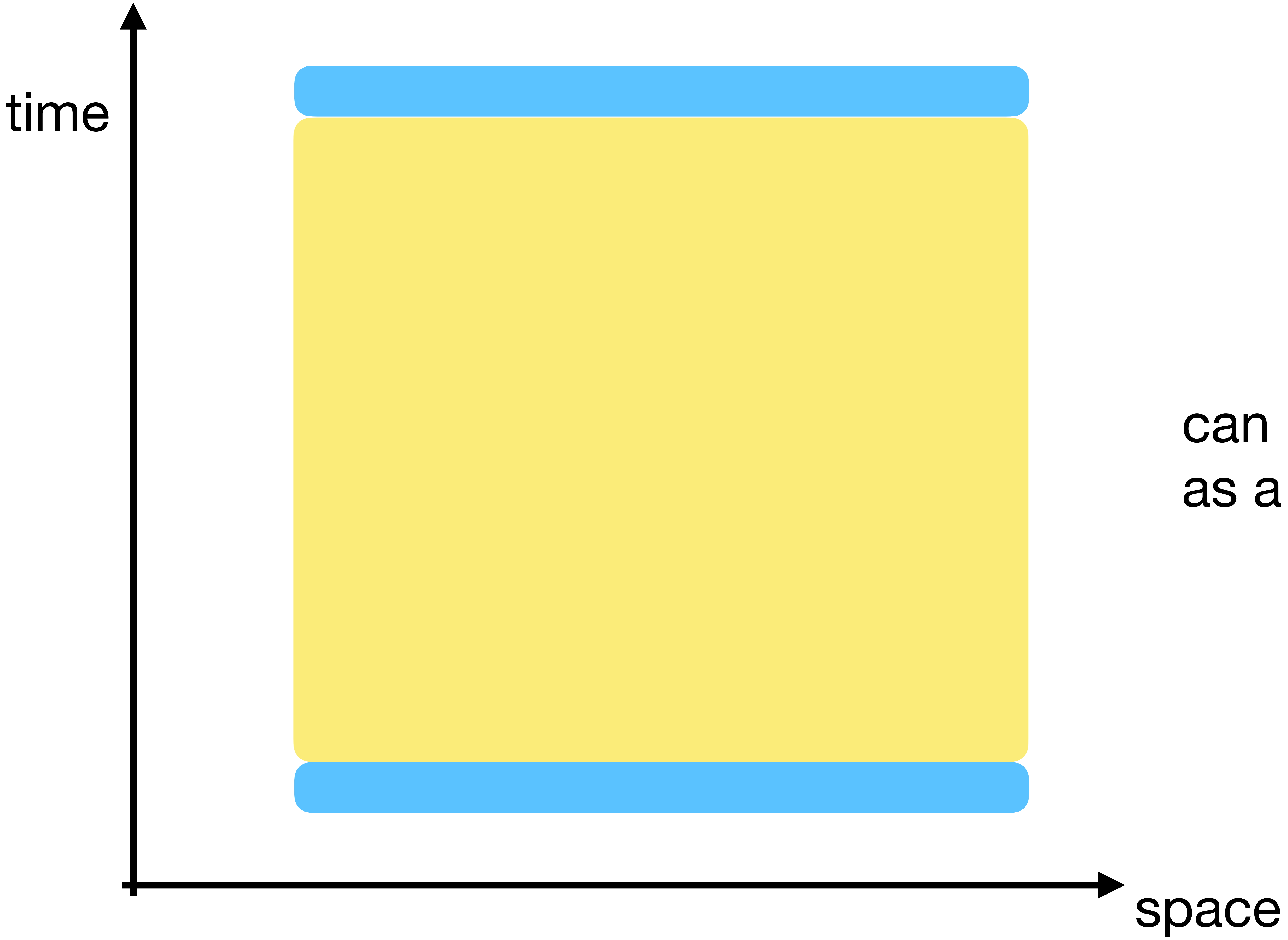


can be seen
as a path integral

Mapping CFT: Loschmidt echo



can be seen
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CFT predictions

Our case <=> Euclidean path integral on infinite strip of width β

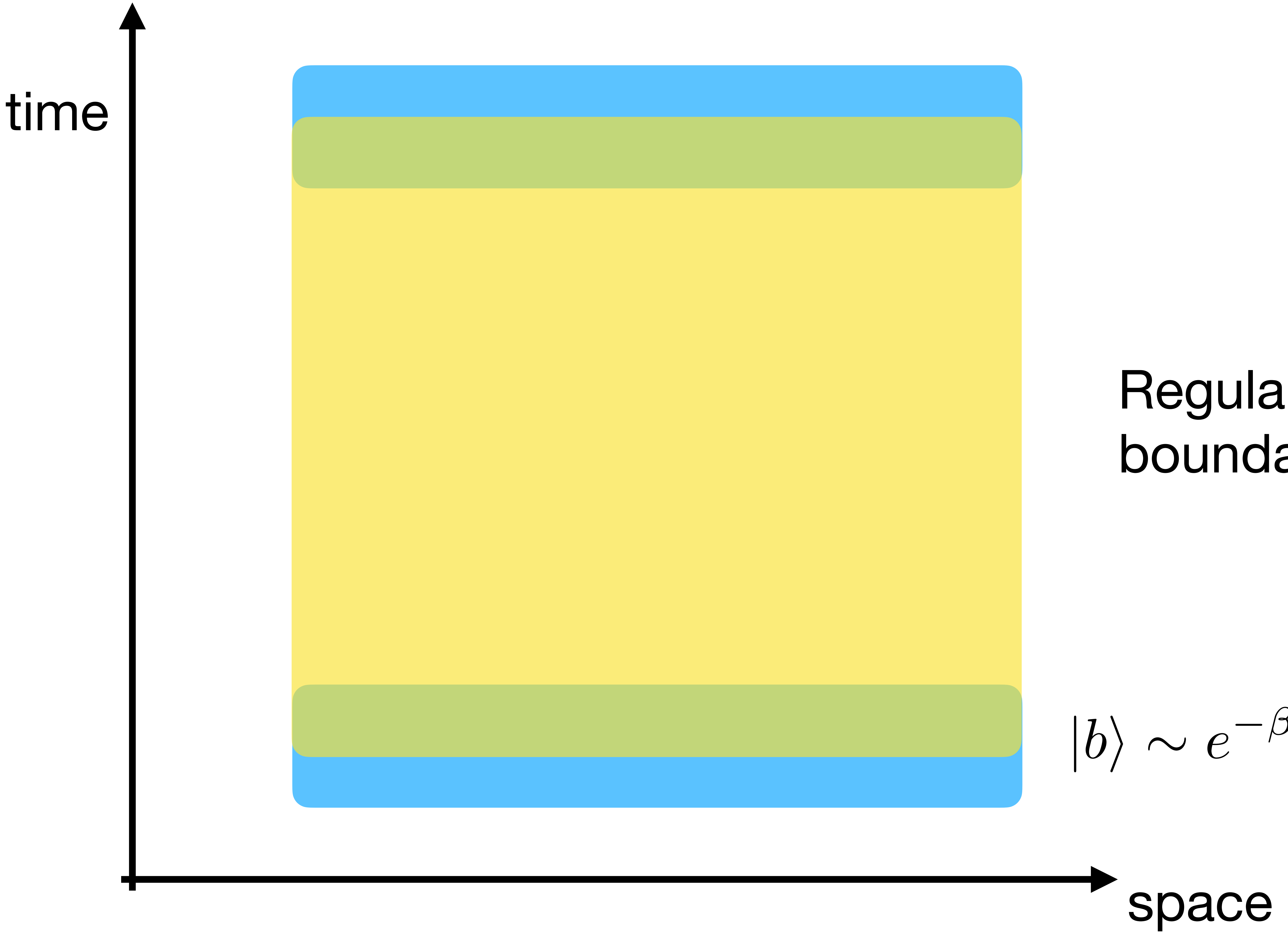
CFT Prediction for Transfer matrix spectrum:

$$\mathcal{T} = \exp\left[-\frac{\kappa_s}{\beta v} + \frac{\pi}{\beta v} L_0 + \mathcal{O}\left(\frac{1}{(\beta v)^2}\right)\right]$$

$$\kappa_s = -\pi c \delta t / 24$$

Central charge

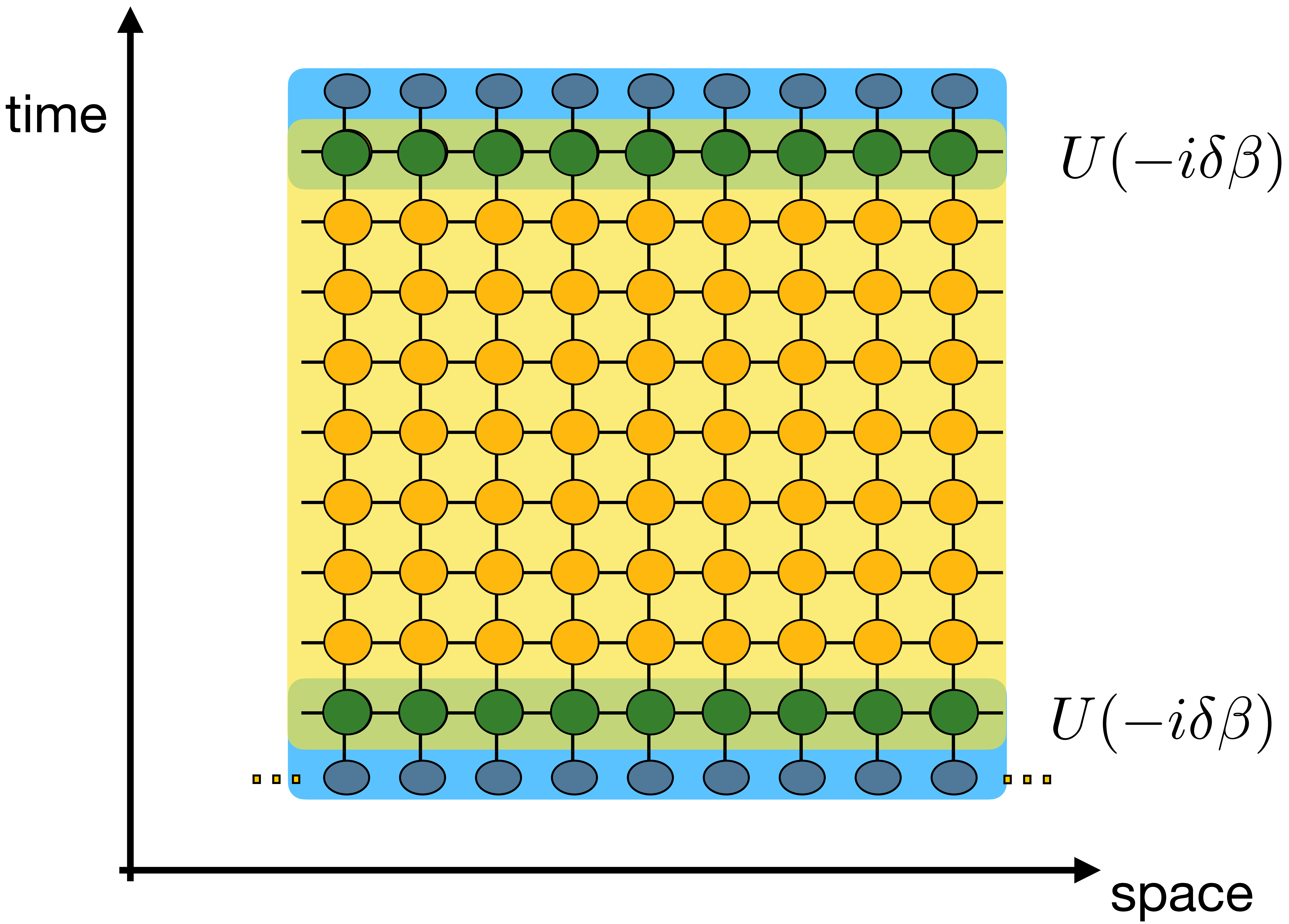
Mapping CFT: Loschmidt echo



Regularization:
boundary states

$$|b\rangle \sim e^{-\beta_0 H} \, |\psi_0\rangle$$

Mapping CFT: Loschmidt echo



CFT predictions

CFT prediction for dominant eigenvalue (including non-universal terms: normalization, boundary effects …)

$$\lambda_0 = \log(-\tau_0) = a\beta v + b + \frac{\kappa}{\beta v}$$

CFT predictions

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~ c

Analytic continuation $\beta = iT + 2\beta_0$

$$Re(\lambda_0) = 2\beta_0 av + b + \dots$$

$$Im(\lambda_0) = avT - \frac{\kappa}{vT} + \dots$$

CFT predictions

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$$Im(\lambda_0) = avT - \frac{\kappa}{vT} + \dots$$

Can compute gaps as well

$$Re(\lambda_1 - \lambda_0) = \mathcal{O}(1/T^2)$$

$$Im(\lambda_1 - \lambda_0) = -\frac{p\pi x_1}{vT} + \dots$$

boundary
critical exponent

CFT predictions

Entropy can be calculated as two-point correlator of twist fields

$$\langle \phi(\omega_1)\phi(\omega_2)\rangle = \left(\frac{\pi}{\beta v}\right)^{2x} \sin\left[\frac{\pi}{\beta v}\left(u_1 - u_2\right)\right]^{-2x}$$

After some algebra and analytic continuation:
generalized entropy along the (temporal) TM

$$S_{gen} = s_0 + \frac{i\pi c}{12} + \frac{c}{6}\log\left[\frac{2T}{\pi}\sin\left(\frac{\pi t}{T}\right)\right]$$

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real part similar to spatial entropy in GS

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Constant imaginary part ~ c

real part similar to spatial entropy in GS

Numerical results: Models

Ising model $$H^{Ising}(g) = -\sum_i \left[\sigma_x^i \sigma_x^{i+1} + g\sigma_z^i\right]$$

Three-state Potts model $$H^{Potts}(g) = -\sum_i \left[\left(\sigma_i \sigma_{i+1}^\dagger + \sigma_i^\dagger \sigma_{i+1}\right) + g(\tau_i + \tau_i^\dagger)\right],$$

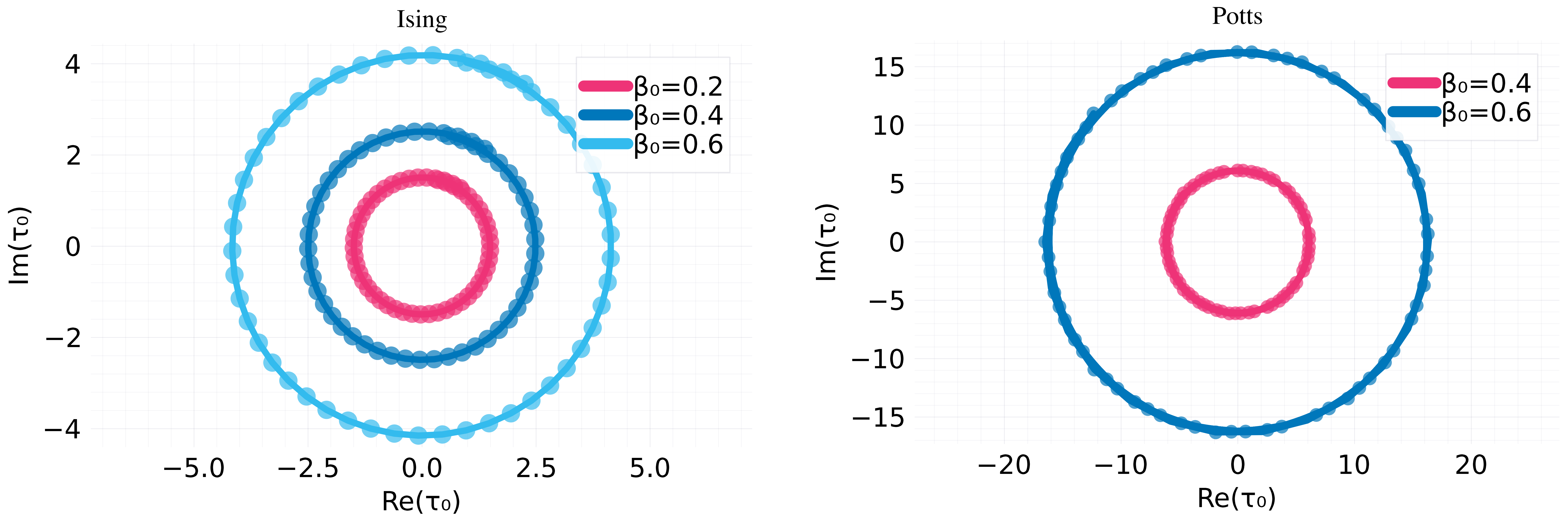
$$\sigma = \sum_{s=0,1,2} \omega^s \left|s\right\rangle \left\langle s\right| , \omega = e^{i2\pi/3} \qquad \tau = \sum_{s=0,1,2} \left|s\right\rangle \left\langle s+1\right|$$

Critical Hamiltonians:
$$H_c^{Ising} \equiv H^{Ising}(g=1)$$
$$H_c^{Potts} = H^{Potts}(g=1)$$

Boundary conditions: free and fixed

Numerical results: dominant eigenvalues

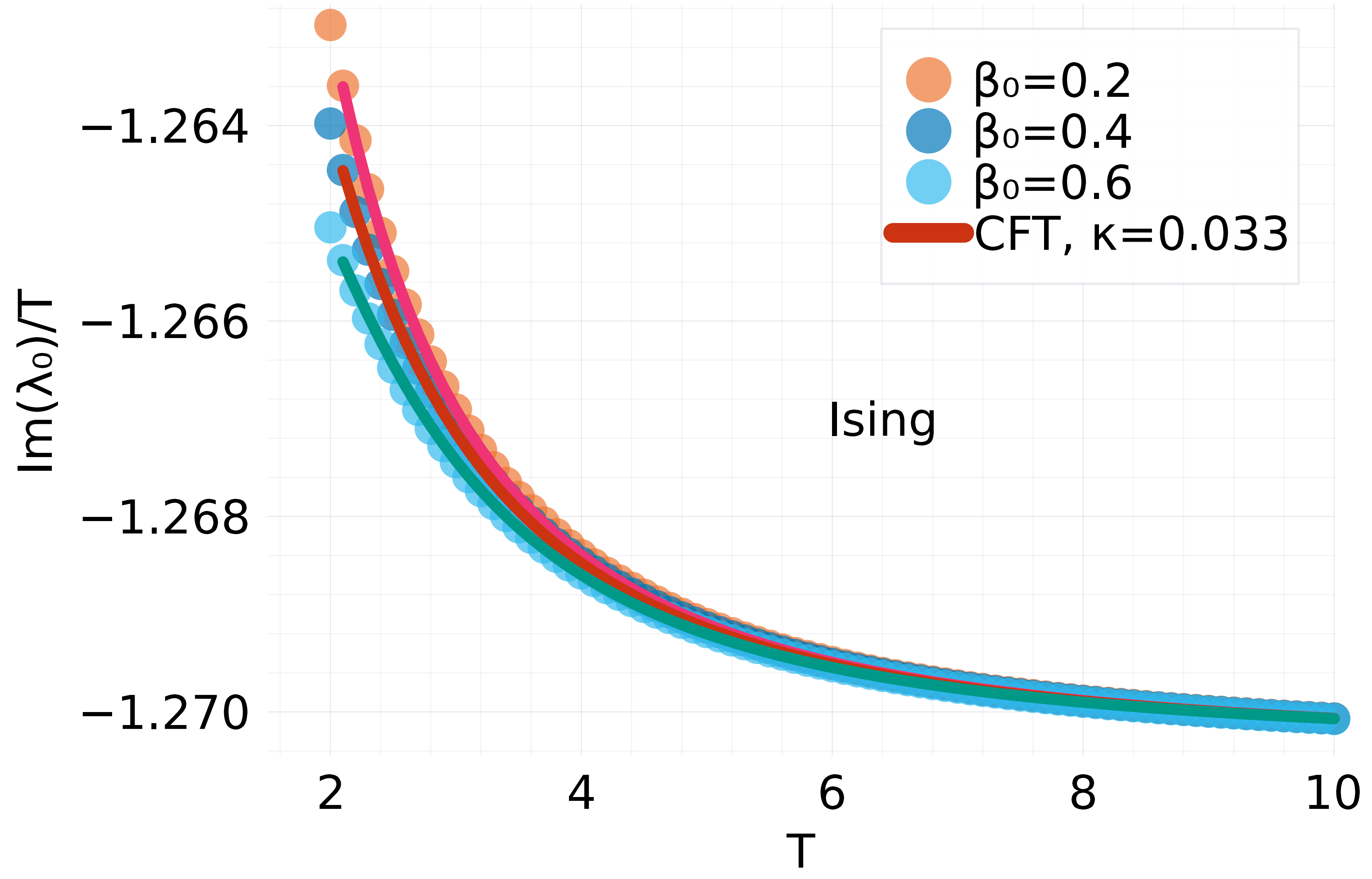
Dominant eigenvalues of Transfer matrix as function of T

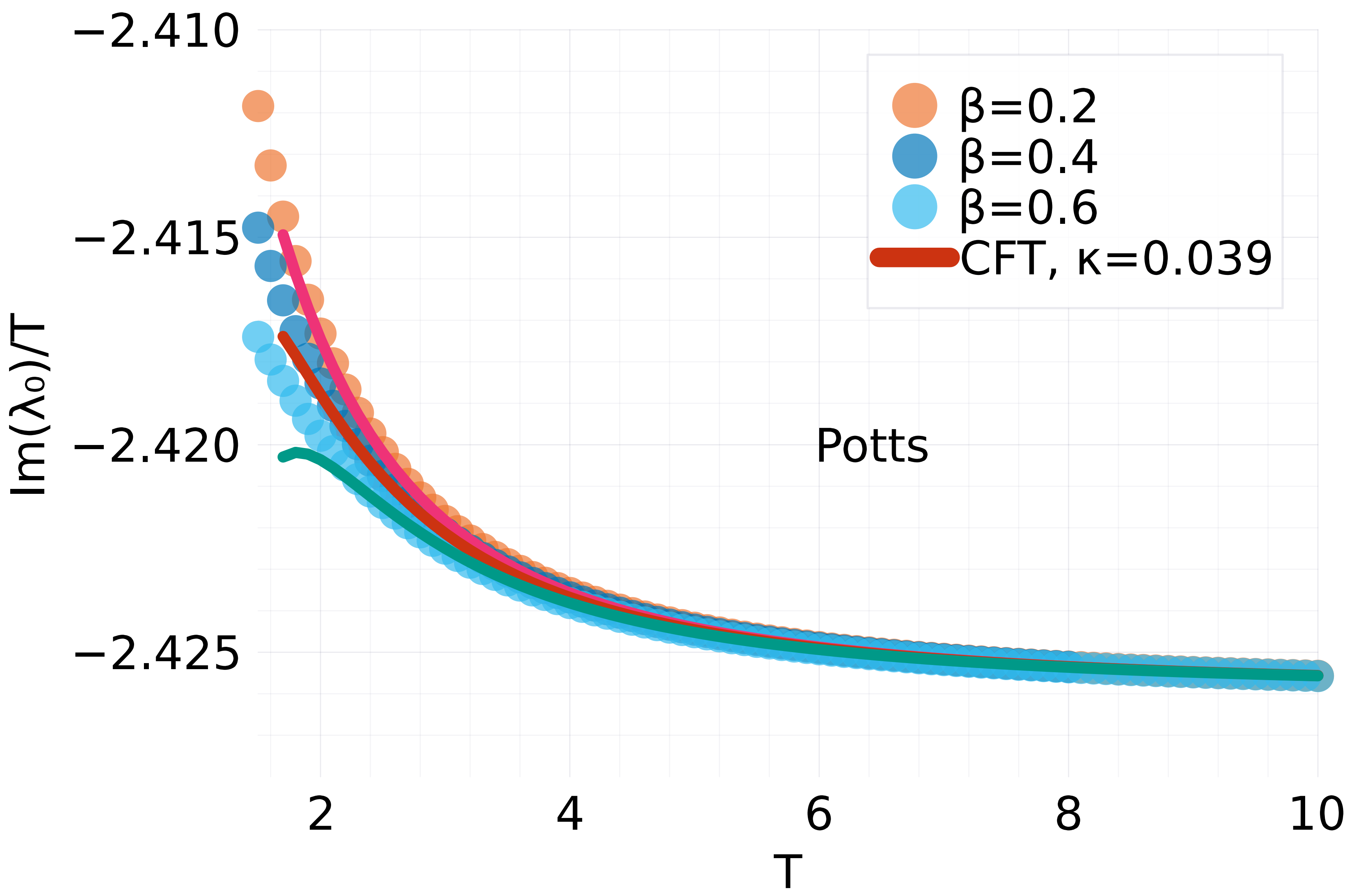


Numerical results:
Imaginary part of dominant eigenvalues

Fit $Im(\lambda_0)/T = a_0 + \kappa/T^2 + a_4/T^4$

(CFT: $Im(\lambda_0) = avT - \frac{\kappa}{vT} + \dots$)





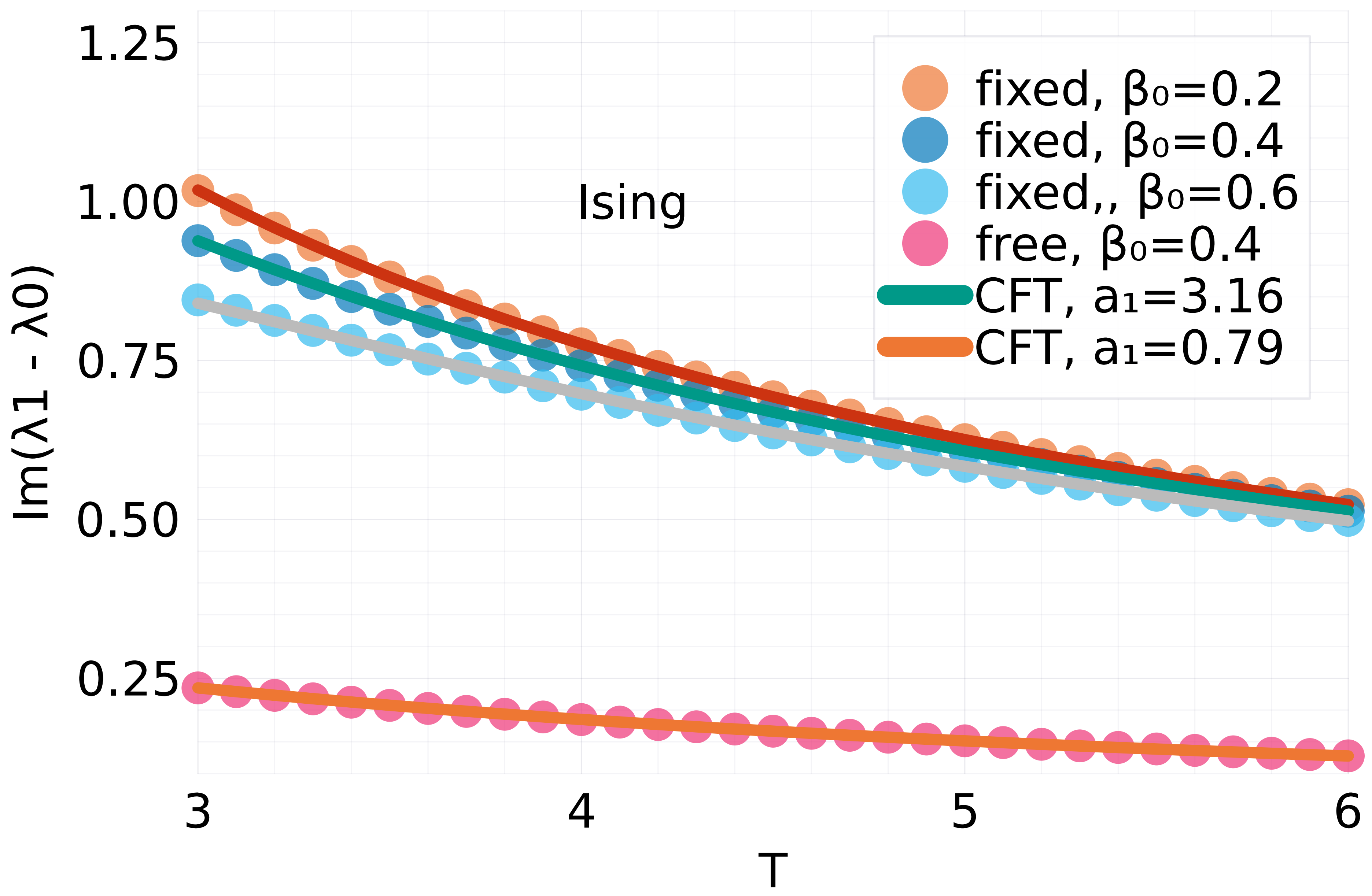
$$c_{fit} \approx 0.49 \approx 1/2 = c_{Ising}$$

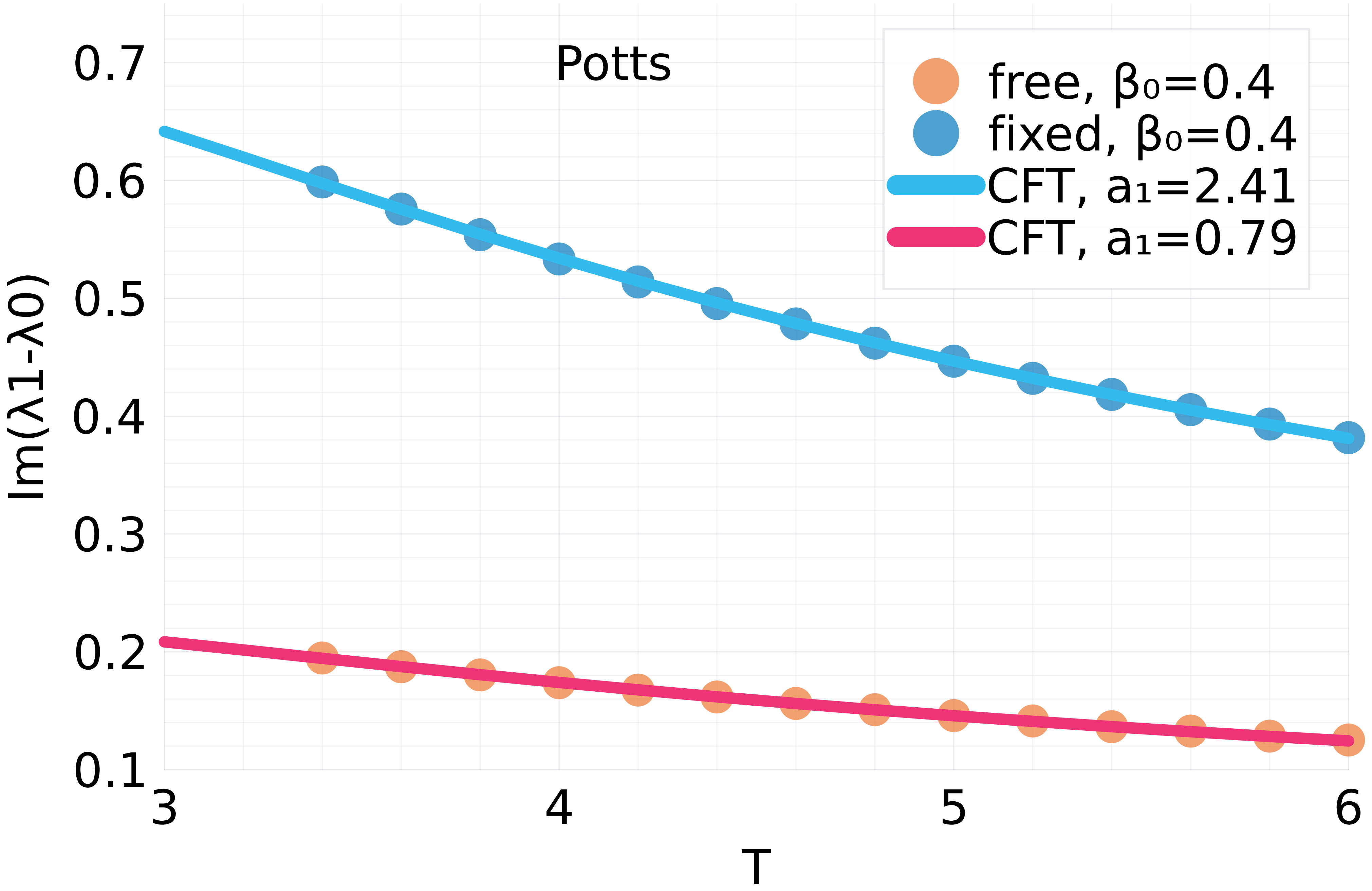
$$c_{fit} \approx 0.78 \approx 4/5 = c_{Potts}$$

Numerical results: Imaginary part of gaps

Fit $Im(\lambda_1 - \lambda_0) = a_1/T + a_3/T^3$

(CFT: $Im(\lambda_1 - \lambda_0) = -\frac{p\pi x_1}{vT} + \dots$)



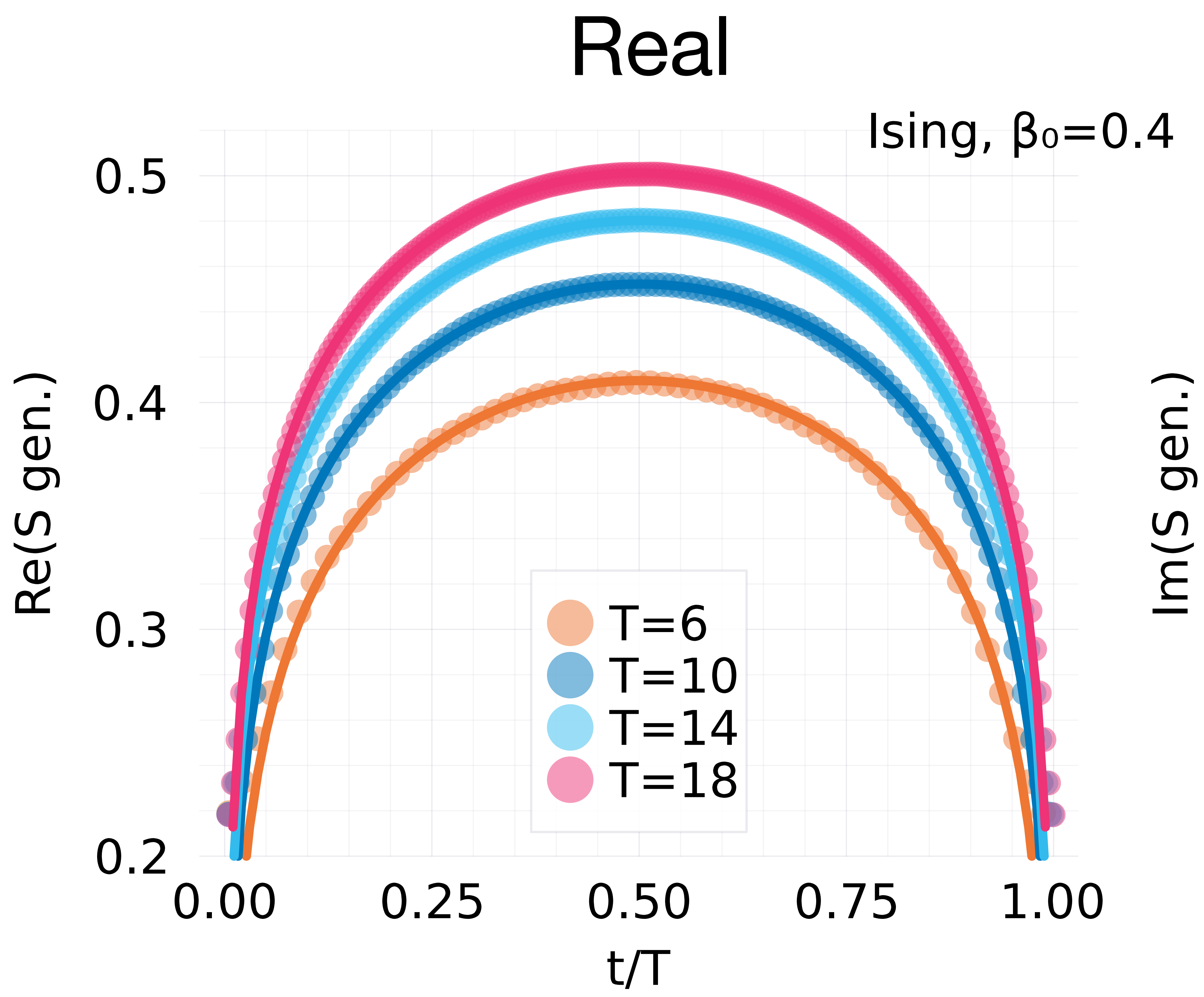


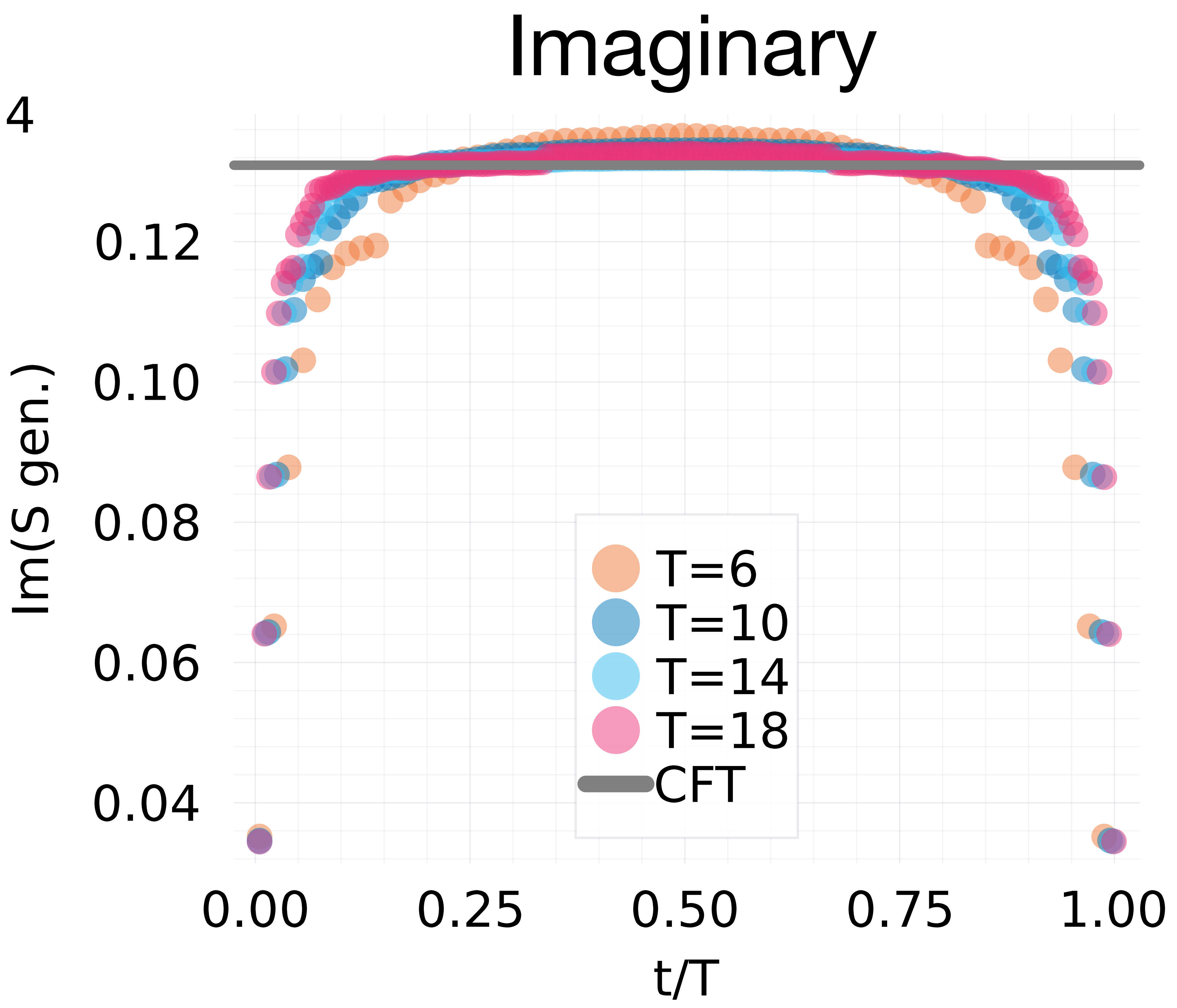
$a_1 = \pi x_1/v = 3.16 \to x_1 \approx 2$
$a_1 = \pi x_1/v = 0.786 \to x_1 \approx 1/2$

$a_1 = 2.41 \to x_1 \approx 0.65 \approx 2/3$
$a_1 = 0.79 \to x_1 \approx 1.98 \approx 2$

Numerical results: entropies

$$S_{gen} = s_0 + \frac{i\pi c}{12} + \frac{c}{6}\log\left[\frac{2T}{\pi}\sin\left(\frac{\pi t}{T}\right)\right]$$





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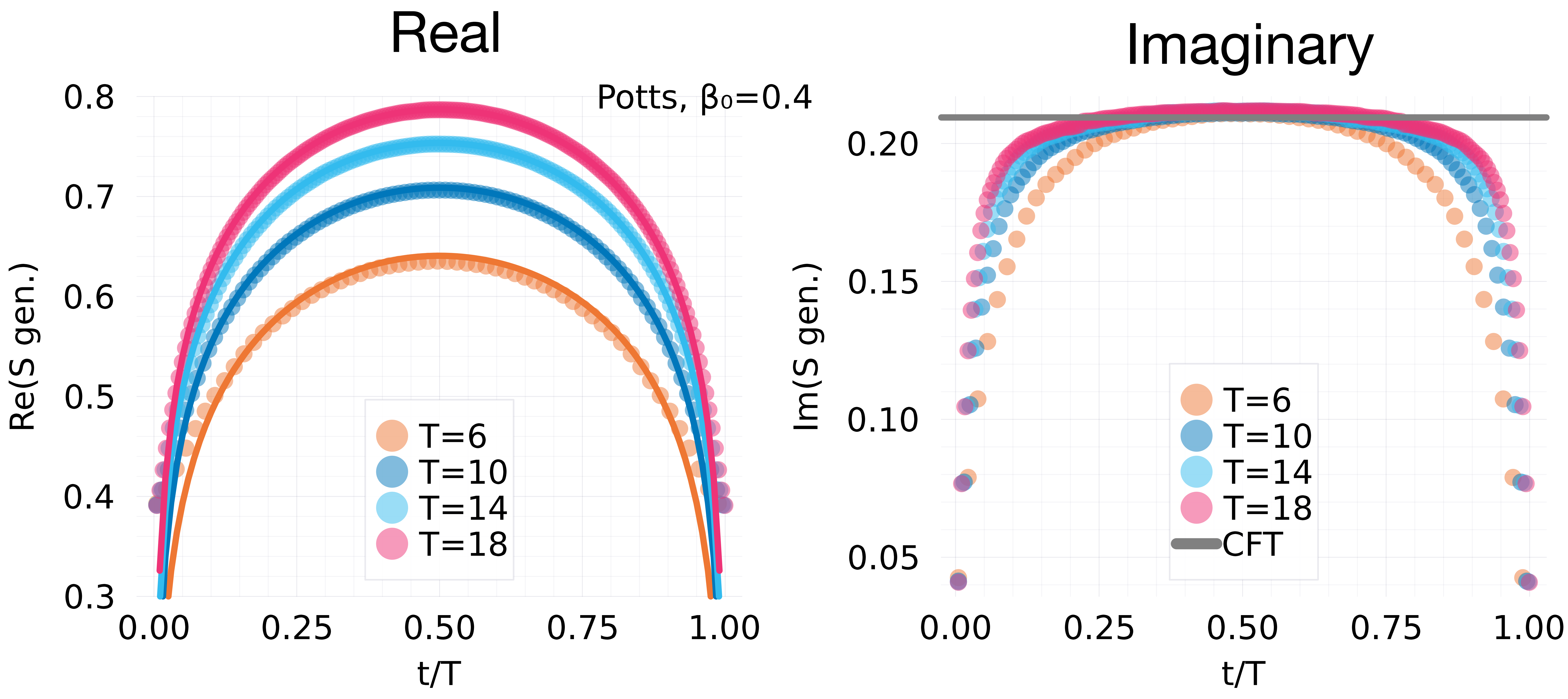

Conclusions

- Time evolution using TN and temporal entanglement

- Loschmidt echo: mapping to path integral allows to make connection to CFT

- CFT predicts spectrum of transfer (transition) matrix
  and generalized temporal entropies

- Logarithmic generalized entanglement growth => simulability with TN

- Gaps closing for large T: Emerging dual unitarity ?