Three- and four-gluon vertices from quenched lattice-QCD

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Outline

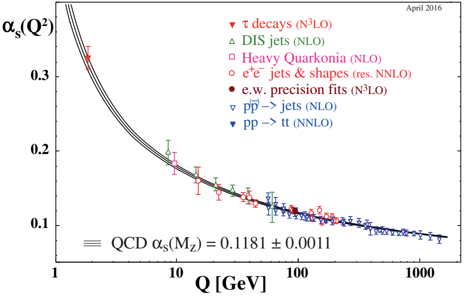
Quantum Chromo-Dynamics

QCD Lagrangian depends on a few parameters: one coupling, α_s, and quark masses (m_u, m_d, m_s, m_c, m_b and m_t).

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a + \sum_{f=u,\cdots,t} \bar{\psi}_f \left(i \not{D} - m_f \right) \psi_f$$

α_s acquires a renormalization scheme dependent running with the momentum.

The running of $\alpha_s(\zeta^2) = \frac{g^2(\zeta^2)}{4\pi}$ is controlled by its RGE, $\frac{d\alpha_s}{d \ln \zeta^2} = \beta(\alpha_s)$



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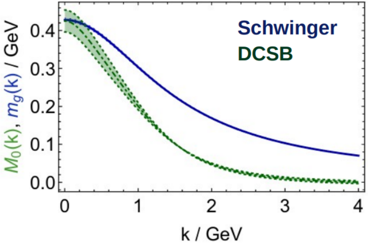
Emergent phenomena:

- Confinement (Hadron masses).
- Dynamically generated gluon-mass.
- Spontaneous chiral symmetry breaking.



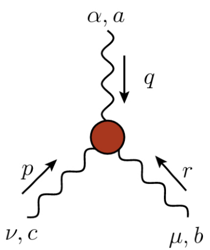
NP approaches:

- Lattice-QCD.
- FUNctional methods.
- QCD vacuum, sum rules,. . .

Gluon self-coupling

Yang-Mills

$$\mathcal{L}_{\rm YM} = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} \; ;$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g \; f^{abc} A^b_\mu A^c_\nu$$



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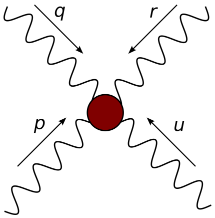
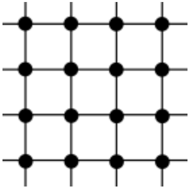
- Gluon self-couplings responsible for the main differences between gluon and photon dynamics.
- Vertices form-factors can be computed from the lattice or DSE.
- Key non-perturbative ingredients in DSE.



Lattice formulation

Path integral in imaginary time:

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int [dUd\psi d\bar{\psi}] \mathcal{O}(U,\psi,\bar{\psi}) e^{-S(U,\psi,\bar{\psi})} \rightarrow \frac{1}{N} \sum_{i=1}^{N} O_i$$



dimensionless; lattice spacing a fixed *a posteriori*.

Pros

- **Just QCD**.
- Regularized per se ($\Lambda \sim a^{-1}$).

Cons

- Finite volume and discretization errors.
- Broken rotational symmetry!
- Expensive chiral fermions.

Lattice setups

Exploited quenched gauge field configurations with:

β	L^4/a^4	a (fm)	confs
5.6	32^4	0.236	2000
5.7	32^4	0.182	1000
5.8	32^4	**0.144**	2000
6.0	32^4	0.096	2000
6.2	32^4	0.070	2000
6.4	32^4	0.054	1300

- Absolute calibration for $\beta = 5.8$ [S. Necco and R. Sommer, Nucl. Phys. B622, 328 (2002)].
- Relative calibrations from gluon propagator scaling [FS *et al.* PRD98 (2018) 114515]

Outline

Computing three-gluon vertex in Landau gauge

Landau gauge

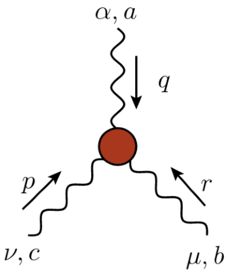
Landau gauge $\partial_\mu A^a_\mu = 0$ fixed numerically, allowing to compute gauge dependent quantities.

- Gluon propagator:

$$\Delta^{ab}_{\mu\nu}(q^2) = \langle \widetilde{A}^a_\mu(q)\widetilde{A}^b_\nu(-q)\rangle = \delta^{ab}\Delta(q^2)P_{\mu\nu}(q)$$

- Three-gluon vertex:

$$\mathcal{G}_{\alpha\mu\nu}(q,r,p) = \frac{f_{abc}\langle \widetilde{A}^a_\alpha(q)\widetilde{A}^b_\mu(r)\widetilde{A}^c_\nu(p)\rangle}{24} \,, \quad q+r+p=0$$



Extracting the transversely projected vertex

From the lattice data, we compute the transversely projected vertex, $\bar{\Gamma}^{\alpha\mu\nu}(q,r,p)$:

$$\mathcal{G}^{\alpha\mu\nu}(q,r,p) = g\bar{\Gamma}^{\alpha\mu\nu}(q,r,p)\,\Delta(q^2)\,\Delta(r^2)\,\Delta(p^2)$$

which corresponds to the transverse projection of the 1PI vertex:

$$\bar{\Gamma}^{\alpha\mu\nu}(q,r,p) = \Gamma^{\alpha'\mu'\nu'}(q,r,p)P^{\alpha}_{\alpha'}(q)P^{\mu}_{\mu'}(r)P^{\nu}_{\nu'}(p)$$

No access to longitudinal part

If the 1PI vertex, $\Gamma^{\alpha\mu\nu}(q,r,p)$ has a longitudinal and a transverse part:

$$\Gamma^{\alpha\mu\nu}(q,r,p) = \Gamma_L{}^{\alpha\mu\nu}(q,r,p) + \Gamma_T{}^{\alpha\mu\nu}(q,r,p)$$

we will only access the transverse one!

Tensorial structure of $\Gamma^{\alpha\mu\nu}(q,r,p)$

The Ball-Chiu decomposition of the 1PI three-gluon vertex has 14 tensors:

$$\ell_1, \ell_2 \ , \cdots \ell_{10}, \qquad t_1, \cdots t_4 \ ,$$

with 10 partially longitudinal and 4 transverse tensors.

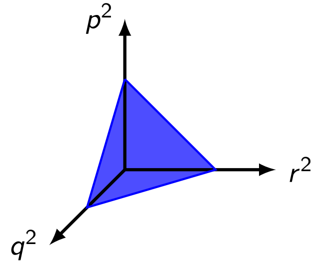
[Phys. Rev. D22 (1980) 2550]

The transversely projected tensor $\bar{\Gamma}^{\alpha\mu\nu}(q,r,p)$ will have at most the contribution of *four* independent tensors:

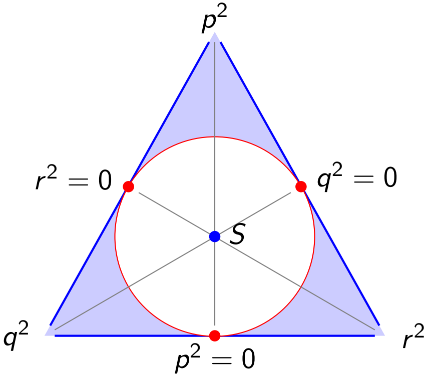
$$\bar{\Gamma}^{\alpha\mu\nu}(q,r,p) = \bar{\Gamma}_1 \lambda_1^{\alpha\mu\nu} + \bar{\Gamma}_2 \lambda_2^{\alpha\mu\nu} + \bar{\Gamma}_3 \lambda_3^{\alpha\mu\nu} + \bar{\Gamma}_4 \lambda_4^{\alpha\mu\nu}$$

Kinematics of the three-gluon vertex

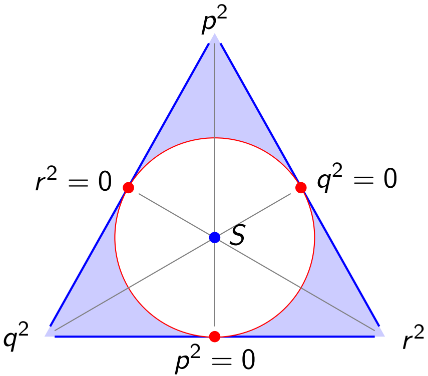
$\bar{\Gamma}^{\alpha\mu\nu}(q, r, p)$ depends on three momenta, with $q + r + p = 0$. The scalar form factors can be cast in terms of the three squared momenta.



Plane with constant $s^2 = \frac{q^2+r^2+p^2}{2}$



Kinematics of the three-gluon vertex



Particular cases:

Case	Def.	$\hat{qr}$	Tensors
Sym.	$q^2 = r^2 = p^2$	$\frac{2\pi}{3}$	$\lambda_{1,2}^{sym}$
Soft gluon	$p = 0$	π	λ_3^{sg}
Collinear	$q = r = -p/2$	0	(none)
Bisectoral	$q^2 = r^2$	$(0, \pi)$	3
General		–	4

Symmetric and soft-gluon cases already studied in [Phys.Lett.B 818 (2021) 136352]

Extraction of form factors

Once we have evaluated $\bar{\Gamma}^{\alpha\mu\nu}(q,r,p)$ from the lattice, we have to solve:

$$\sum_i \bar{\Gamma}_i(q^2,r^2,p^2)\ \tilde{\lambda}_i^{\alpha\mu\nu}(q,r,p)\tilde{\lambda}_{j\,\alpha\mu\nu}(q,r,p) = \bar{\Gamma}^{\alpha\mu\nu}(q,r,p)\ \tilde{\lambda}_j^{\alpha\mu\nu}(q,r,p)$$

or equivalently obtain a projector $\tilde{P}_i(q,r,p)$:

$$\bar{\Gamma}_i(q^2,r^2,p^2) = \bar{\Gamma}^{\alpha\mu\nu}(q,r,p)\tilde{P}_{i\,\alpha\mu\nu}(q,r,p)$$

Renormalization: momentum subtraction renormalization schemes

Propagator

$$\Delta_{R,\zeta^2}(p^2) = \lim_{a\to 0} Z_A^{-1}(\zeta^2, a)\Delta_0(p^2, a)$$

with

$$\Delta_{R,\zeta^2}(p^2)\Big|_{p^2=\zeta^2} = \frac{1}{\zeta^2}$$

From the lattice, we set

$$Z_A(\zeta^2, a) = \zeta^2 \Delta_0(\zeta^2, a)$$

Renormalization: momentum subtraction renormalization schemes

Three-gluon vertex

$$\overline{\Gamma}_{R,\zeta^2}(q^2,r^2,p^2)\Big|_{k(q^2,r^2,p^2)} = \lim_{a\to 0} Z_3(\zeta^2,a)\overline{\Gamma}_0(q^2,r^2,p^2;a)$$

with

$$\overline{\Gamma}_{Rk,\zeta^2}(q^2,r^2,p^2)\Big|_{k(\zeta^2)} = 1$$

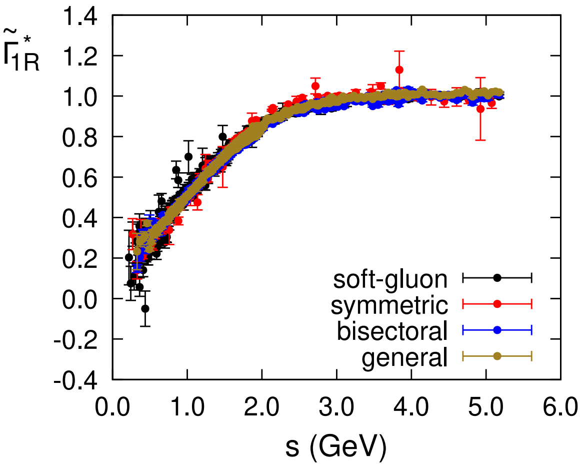
for some kinematic configuration defined by $k(\zeta^2) \equiv (\zeta^2, \theta_{qp}, \theta_{rp})$ corresponding to a kinematic point lying on the plane $\frac{q^2+r^2+p^2}{2} = \zeta^2$ and the two angles among momenta θ_{qp}, θ_{rp}.

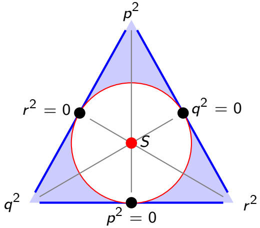
From the lattice, we set

$$Z_{3k}^{-1}(\zeta^2,a) = \overline{\Gamma}_0(q^2,r^2,p^2)\Big|_{k(\zeta^2)}$$

Results for the general kinematics: planar degeneracy.

Tree-level form-factor renormalized in **soft-gluon** scheme at $\zeta = 4.3$ GeV.

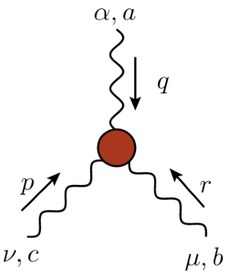




- Nice scaling in terms of $s^2 = \frac{q^2+r^2+p^2}{2}$ for different kinematics!

[F. Pinto-Gomez, FS, *et al*, PhysRevD.110.014005]

Conclusions from three-gluon vertex.



The rest of form factors are subdominant or completely negligible, and the tree-level one, $\tilde{\lambda}_{tl}^{\alpha\mu\nu}(q, r, p)$, dominates.

Planar degeneracy

The full vertex seems to be well described by:

$$\bar{\Gamma}^{\alpha\mu\nu}(q, r, p) \approx \bar{\Gamma}^{sg}(s^2)\Big|_{s^2=\frac{q^2+r^2+p^2}{2}} \tilde{\lambda}_{tl}^{\alpha\mu\nu}(q, r, p)$$

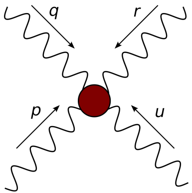
[F. Pinto-Gomez, FS, *et al*, PhysRevD.110.014005] & JRQ talk!

Outline

Four-gluon vertex.

Four-point Green function depends on four momenta:

$$\mathcal{G}^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u) \equiv \langle \widetilde{A}^a_\alpha(q)\widetilde{A}^b_\beta(r)\widetilde{A}^c_\gamma(p)\widetilde{A}^d_\delta(u)\rangle$$



Tensorial structure:

- [Color] $\delta^{ab}\delta^{cd}$ (3), $f^{abe}f^{cde}$ (3), $d^{abe}d^{cde}$ (3), $f^{abe}d^{cde}$ (6).
- [Lorentz] $g_{\alpha\beta}g_{\gamma\delta}$ (3), $g_{\alpha\beta}q_\gamma q_\delta$ (54), $q_\alpha q_\beta q_\gamma q_\delta$ (81).
- In Landau gauge there are 41 Lorentz independent tensors!

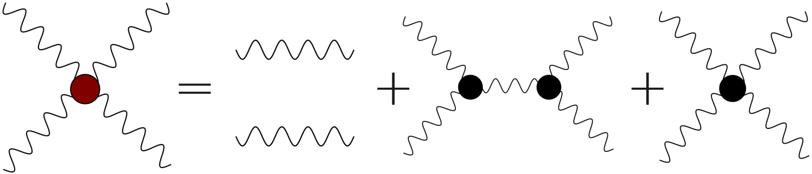
Kinematics:

- Six Lorentz invariants: (q^2, r^2, p^2 and u^2 do not fix the kinematics)

[G. Eichmann, *et al.*, Phys.Rev.D92 (2015) 056006]

One-particle-irreducible four-gluon vertex.

Depending on the kinematics, four-gluon Green function can have contributions from disconnected or three-gluon diagrams.



We are interested in the 1PI four-gluon vertex, so we have to avoid or subtract the disconnected and 3g diagrams.

Special kinematical configurations.

Four-gluon vertex hard to extract from the lattice:

- Complex tensorial structure
- Disconnected or 3g diagrams
- Noise from 4-point Green function

Recent lattice data at [M. Colaco *et al.* Phys. Rev. D109 (2024) 074502] & Orlando's talk

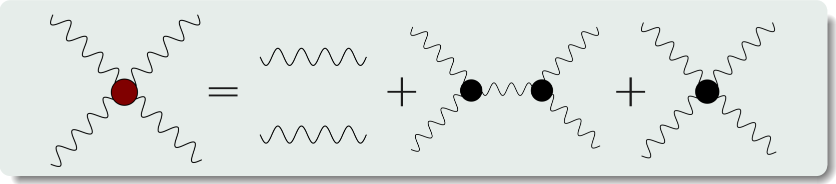
Restricted kinematics

For parallel momenta $\left(p, \alpha p, \beta p, (-1-\alpha-\beta)p\right)$ the number of tensors is much smaller and one can eliminate disconnected and 3g diagrams.

Collinear kinematics.

With parallel momenta we will focus in the special kinematics:

- $(p, p, -p, -p)$
- $(p, \alpha p, -p, -\alpha p)$
- $(p, p, p, -3p)$ $*$
- $(p, \alpha p, -p - \alpha p, 0)$



- Disconnected contributions dominate. The signal is the same size of the systematic errors.
- No transverse 3g vertex with collinear momenta.

$*$ Widely studied, since [D. Binosi *et al.* JHEP9 (2014), 59] till [M. Colaco *et al.* Phys. Rev. D109 (2024) 074502]

Collinear kinematics.

The transversely projected vertex, $\bar{\Gamma}^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u)$ is obtained as:

$$g\bar{\Gamma}^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u) = \frac{\mathcal{G}^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u)}{\Delta(q^2)\Delta(r^2)\Delta(p^2)\Delta(u^2)} = \sum_i g\bar{\Gamma}_i \lambda_i{}^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u)$$

We will be interested in the tree-level form-factor corresponding to the tensor

$$\begin{aligned}
(\lambda_{tl})^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u) \;=\; & \Big[-f^{abe}f^{cde}(g_{\alpha'\gamma'}g_{\beta'\delta'} - g_{\alpha'\delta'}g_{\beta'\gamma'}) \\
- \;& f^{ace}f^{bde}(g_{\alpha'\beta'}g_{\gamma'\delta'} - g_{\alpha'\delta'}g_{\beta'\gamma'}) \\
- \;& f^{ade}f^{bce}(g_{\alpha'\beta'}g_{\gamma'\delta'} - g_{\alpha'\gamma'}g_{\beta'\delta'}) \Big] \\
\times \;& P^{\alpha'}_{\alpha}(q)P^{\beta'}_{\beta}(r)P^{\gamma'}_{\gamma}(p)P^{\delta'}_{\delta}(u)
\end{aligned}$$

Collinear kinematics.

With parallel momenta there are 15 independent tensors [Aguilar *et al*, Eur.Phys.J.C 84 (2024) 7, 676], and one can use a projector to isolate the tree-level form-factor:

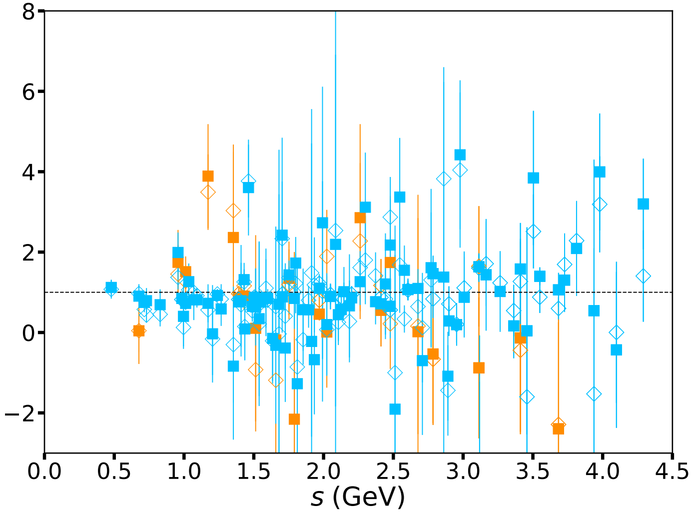
$$g\bar{\Gamma}_0 = g\bar{\Gamma}^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u) \otimes (\mathcal{P}_{tl})^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u)$$

that is in principle different from the contraction

$$g\,\bar{G}_0 = \frac{g\bar{\Gamma}^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u) \otimes (\lambda_{tl})^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u)}{(\lambda_{tl})^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u) \otimes (\lambda_{tl})^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u)}$$

because the 15 tensors are not orthogonal.

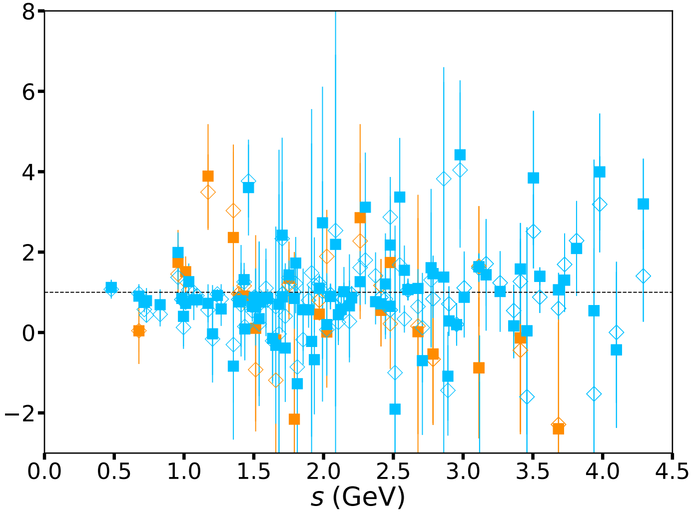
Collinear kinematics.



- $(p, p, p, -3p)$
- $(p, \alpha p, -p - \alpha p, 0)$

- Full symbols: $\bar{\Gamma}_0$.
- Empty symbols: $\bar{G}_0$.

Collinear kinematics: summary.



The tree-level form-factor $\overline{\Gamma}_0$ is:

- compatible with the contraction $\overline{G}_0$
- still rather noisy
- vaguely compatible with its tree-level value $\overline{\Gamma}_0 = 1$

Soft-gluon kinematics

A different strategy: non-parallel momenta with one soft gluon!

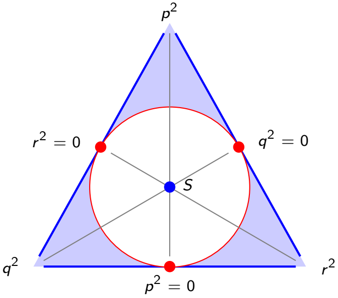
$$\Big(p, q, r, 0\Big)$$

Pros:

- 3g diagrams easily subtracted from lattice data (for each set of momenta!).
- Lots of lattice momenta (Same kinematics as three-gluon).

Cons:

- Full set of tensors contribute.
- Projection not available. We compute the contraction with tree-level tensor: $\bar{G}_0$.

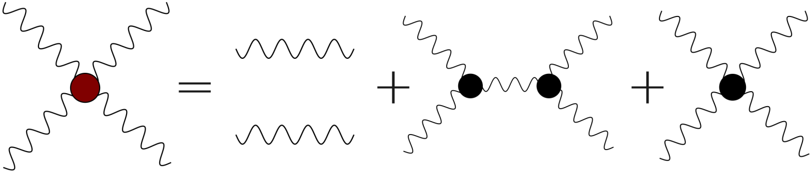


Four-gluon vertex (preliminary)

We perform the contraction with the tree-level tensor

$$g\,\bar{G}_0 = \frac{g\bar{\Gamma}^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u) \otimes (\lambda_{tl})^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u)}{(\lambda_{tl})^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u) \otimes (\lambda_{tl})^{abcd}_{\alpha\beta\gamma\delta}(q,r,p,u)}$$

that in this case will contain the contribution from three-gluon diagrams apart from the four-gluon 1PI vertex:



For each set of lattice momenta $(q, r, p, 0)$, we compute the three-gluon diagram using the lattice data for gluon propagator and 3g vertex and subtract it.

Renormalization

Four-gluon vertex

$$\overline{\Gamma}_{R,\zeta^2}(q,r,p,u)\Big|_{k(\zeta^2)} = \lim_{a\to 0} Z_4(\zeta^2,a)\overline{\Gamma}_0(q,r,p,u;a)$$

with

$$\overline{\Gamma}_{Rk,\zeta^2}(q,r,p,u)\Big|_{k(\zeta^2)} = 1$$

for some kinematic configuration defined by $k(\zeta^2)$.

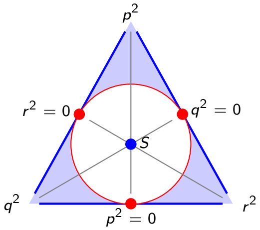
From the lattice, we set

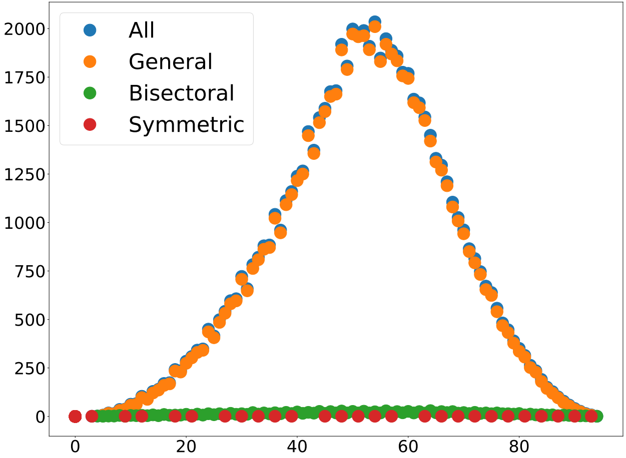
$$Z_{4k}^{-1}(\zeta^2,a) = \overline{\Gamma}_0(q,r,p,u)\Big|_{k(\zeta^2)}$$

with $k(\zeta^2)$ defined by $\zeta^2 = \frac{q^2+r^2+p^2+u^2}{2}$.

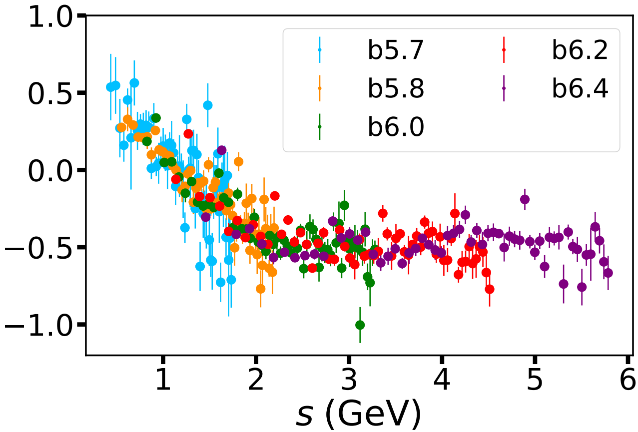
Planar degeneracy

There is a large number of lattice momenta $(q, r, p, 0)$ with $p + r + p = 0$, but each one is too noisy, so we average separately **assuming planar degeneracy**.





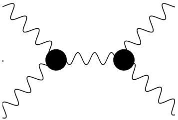
Four-gluon vertex (preliminary)



- Nice scaling for different β's
- Unsubstracted Green function negative for large-s

Four-gluon vertex (preliminary)

To isolate the 1PI vertex, we need to subtract the 3g diagrams (including crossed terms):



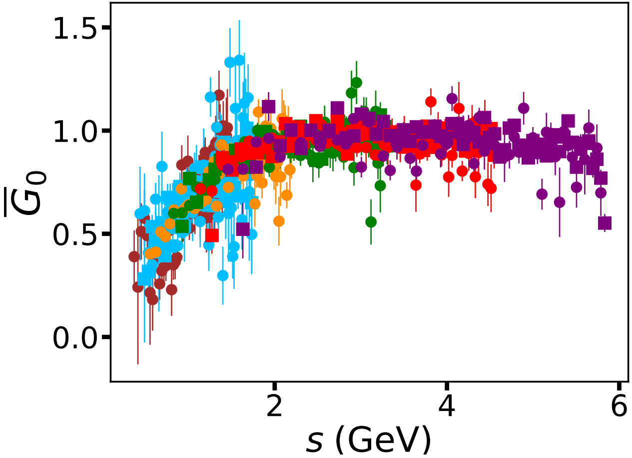
$$\otimes \frac{(\lambda_{tl})^{\alpha\beta\gamma\delta}_{abcd}}{(\lambda_{tl})^{abcd}_{\alpha\beta\gamma\delta} \otimes (\lambda_{tl})^{\alpha\beta\gamma\delta}_{abcd}} = f(q^2, r^2, p^2)\,\Delta(p^2)\,\overline{\Gamma}^{\rm sg}(p^2)\overline{\Gamma}(q^2, r^2, p^2)$$

where $\overline{\Gamma}^{sg}$ is the soft-gluon tree-level form-factor, and

$$f(x,y,z) = \frac{(5(x+y)+z)\left(x^2 - 2x(y+z) + (y-z)^2\right)}{72xy}\,.$$

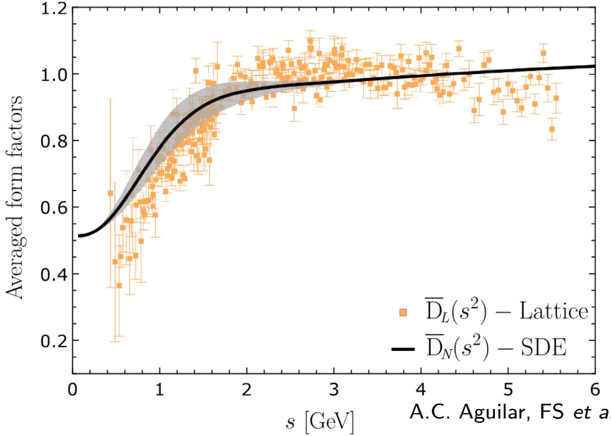
We assume the three-gluon vertex is dominated by tree-level form-factor.

Four-gluon vertex (preliminary)



- After subtracting the 3g diagrams the 4g form-factor becomes positive.
- An IR finite supreession appears, qualitatively similar to:
  - M. Huber, Phys.Rev.D 101 (2020)
  - N. Barrios, *et al*, Phys. Rev. D 109 (2024), L091502
  - Aguilar *et al*, Eur.Phys.J.C 84 (2024) 7, 676
  - ...

Comparison of the results obtained from the lattice with the ones coming from the Schwinger-Dyson equation governing this vertex reveals a nice agreement between both approaches!



A.C. Aguilar, FS *et al* 2408.06135 & LR talk!

Outline

Three-gluon coupling

MOM three-gluon strong coupling

$$\begin{aligned}\mathcal{G}(q^2, r^2, p^2; a)\big|_0 &= \langle \widetilde{A}(q)\widetilde{A}(r)\widetilde{A}(p)\rangle \\ &= g_0(a)\overline{\Gamma}_0(q^2, r^2, p^2; a)\Delta_0(q^2; a)\Delta_0(r^2; a)\Delta_0(p^2; a)\end{aligned}$$

Implies:

$$g_{Rk}(\zeta^2) = \lim_{a\to 0} Z_{3k}^{-1}(\zeta^2; a) Z_A^{3/2}(\zeta^2; a) g_0(a)$$

Finally,

$$\alpha_{Rk}(\zeta^2) = \frac{g_{Rk}^2(\zeta^2)}{4\pi}$$

Planar degeneracy at action...

If we choose soft-gluon kinematics, $s(\zeta^2)$ defined by $q^2 = r^2$ and $p^2 = 0$, we have

$$s(\zeta^2) \equiv (\zeta^2, \pi/2, \pi/2)$$

we recover the soft-gluon MOM scheme definition widely used in the past with:

$$Z_{3s}^{-1}(\zeta^2, a) = \overline{\Gamma}_0(\zeta^2, \zeta^2, 0; a) \equiv \overline{\Gamma}_0^{sg}(\zeta^2; a)$$

For any other scheme k,

$$g_{Rk}(\zeta^2) = \frac{\overline{\Gamma}_0(q^2, r^2, p^2; a)\big|_{k(\zeta^2)}}{\overline{\Gamma}_0^{sg}(\zeta^2; a)} g_{Rs}(\zeta^2) = \overline{\Gamma}_{Rs(\zeta^2)}(q^2, r^2, p^2; a)\big|_{k(\zeta^2)} \, g_{Rs}(\zeta^2)$$

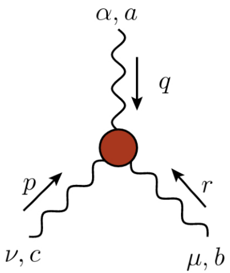
Planar degeneracy at action...

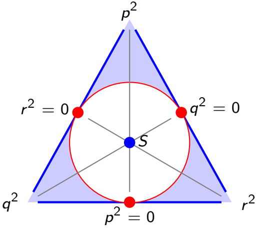
If planar degeneracy holds,

$$\overline{\Gamma}_{Rs(\zeta^2)}(q^2, r^2, p^2; a)\Big|_{k(\zeta^2)} = 1$$

and thus:

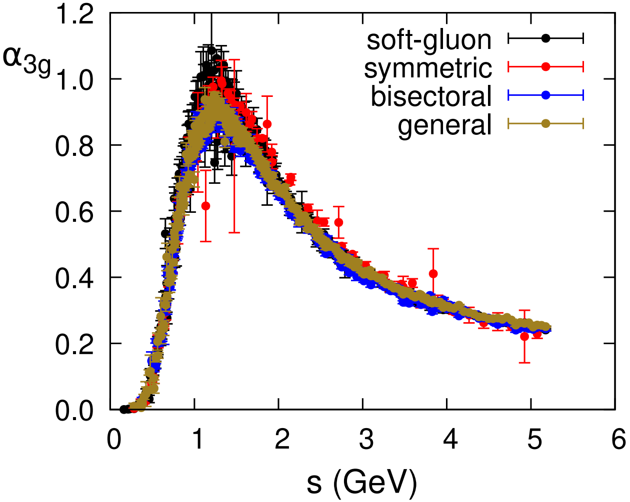
$$g_{Rk}(\zeta^2) = g_{Rs}(\zeta^2)$$

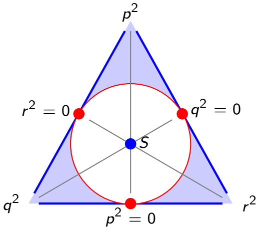




Each kinematic configuration defines a different renormalization scheme $k(\zeta^2)$.

Three-gluon coupling





- Nice scaling!
- Suggest a universal 3g coupling...

[PhysRevD.110.014005]

JRQ talk!

Four-gluon coupling

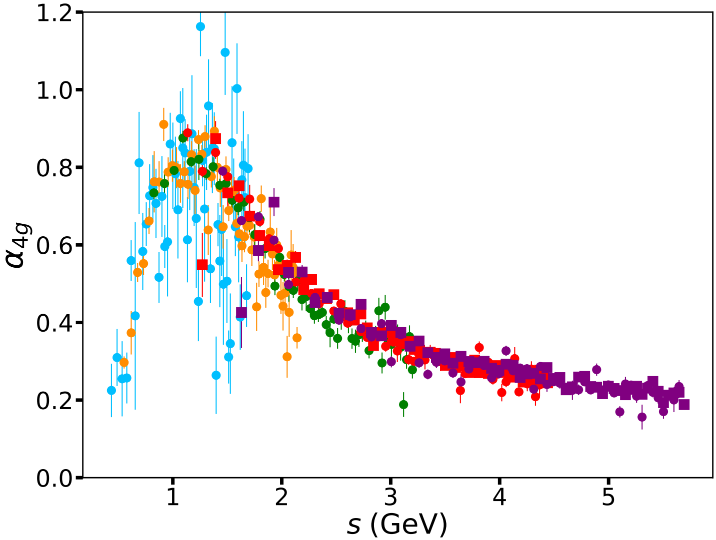
MOM four-gluon strong-coupling

$$\begin{aligned}\mathcal{G}(p_1,p_2,p_3,p_4;a)\big|_0 &= \langle \widetilde{A}(p_1)\widetilde{A}(p_2)\widetilde{A}(p_3)\widetilde{A}(p_4)\rangle \\ &= g_0^2(a)\overline{\Gamma}_0(p_1,p_2,p_3,p_4;a)\prod_{i=1}^{4}\Delta_0(p_i^2;a)\end{aligned}$$

Implies:

$$g_{Rk}^2(\zeta^2) = \lim_{a\to 0} Z_{4k}^{-1}(\zeta^2;a)Z_A^2(\zeta^2;a)g_0^2(a)$$

Four-gluon coupling (preliminary)



- Nice scaling for the different β's.
- Running-coupling similar to α_{3g}.

Outline

Conclusions

- First results for 4g vertex beyond collinear kinematics.
- Four-gluon vertex shows IR suppression in agreement with SDE.
- New definition of a general MOM renormalization scheme for the 3g vertex exploiting *planar degeneracy*.
- First lattice calculation of α_{4g}.