THE CENTER-SYMMETRIC LANDAU GAUGE MEETS THE LATTICE

Urko Reinosa*

(together with D. M. van Egmond, O. Oliveira, J. Serreau, P. Silva and M. Tissier)

*Centre de Physique Théorique, Ecole Polytechnique,
CNRS & Institut Polytechnique de Paris

QuantFunc24, Valencia, September 2024.

Strong interest for the evaluation of gluon correlation functions both from the lattice or from the various continuum approaches:

$$\langle A\rangle, \quad \langle AA\rangle, \quad \langle AAA\rangle\,, \quad \ldots$$

Strong interest for the evaluation of gluon correlation functions both from the lattice or from the various continuum approaches:

$$\langle A \rangle, \quad \langle AA \rangle, \quad \langle AAA \rangle\,, \quad \ldots$$

These are gauge-dependent building blocks from which one aims at evaluating observables $\langle \mathcal{O}[A] \rangle$ and study the physics.

Strong interest for the evaluation of gluon correlation functions both from the lattice or from the various continuum approaches:

$$\langle A\rangle, \quad \langle AA\rangle, \quad \langle AAA\rangle\,, \quad \dots$$

These are gauge-dependent building blocks from which one aims at evaluating observables $\langle\mathcal{O}[A]\rangle$ and study the physics.

Question: is it possible to access some physical features
more directly from the correlation functions?

Strong interest for the evaluation of gluon correlation functions both from the lattice or from the various continuum approaches:

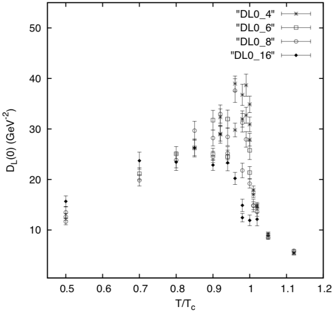
$$\langle A \rangle, \quad \langle AA \rangle, \quad \langle AAA \rangle\,, \quad \dots$$

These are gauge-dependent building blocks from which one aims at evaluating observables $\langle \mathcal{O}[A] \rangle$ and study the physics.

Question: is it possible to access some physical features more directly from the correlation functions?

Can the correlation functions probe the breaking of center symmetry in Yang-Mills theories at finite temperature and thus the confinement/deconfinement transition?

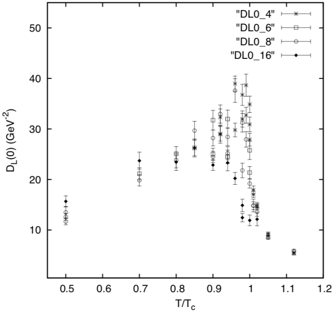
This is an old question ...



Cucchieri and Mendes, PoS 183 (2014).

Cucchieri et al (2007); Fischer et al (2010); Bornyakov et al (2011); Fister et al (2011); Aouane et al (2012); Maas et al (2012); Silva et al (2013) & (2016).

This is an old question ...



Cucchieri and Mendes, PoS 183 (2014).

Cucchieri et al (2007); Fischer et al (2010); Bornyakov et al (2011); Fister et al (2011); Aouane et al (2012); Maas et al (2012); Silva et al (2013) & (2016).

... to which we would like to contribute with some recent developments.

This is in fact the continuation of some work that D. M. van Egmond and myself presented @FunQCD, Valencia, 2022.

This is in fact the continuation of some work that D. M. van Egmond and myself presented @FunQCD, Valencia, 2022.

There, we argued that the Landau gauge correlators are maybe not the most appropriate for probing the breaking of center symmetry.

This is in fact the continuation of some work that D. M. van Egmond and myself presented @FunQCD, Valencia, 2022.

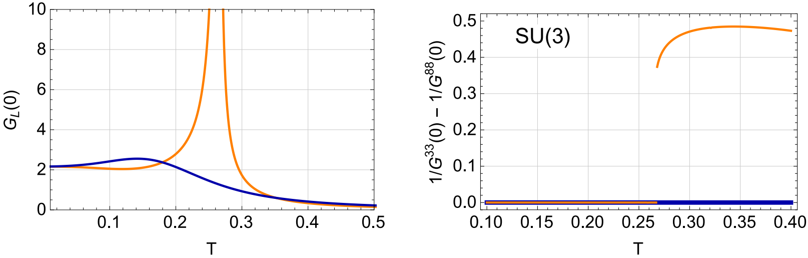
There, we argued that the Landau gauge correlators are maybe not the most appropriate for probing the breaking of center symmetry.

We also introduced a new gauge, the center-symmetric Landau gauge, that allows one to construct order parameters directly from the gauge-field correlators.

We illustrated the benefits of this gauge by showing some explicit calculations of the one- and two-point functions $\langle A \rangle$ and $\langle AA \rangle$.

We illustrated the benefits of this gauge by showing some explicit calculations of the one- and two-point functions $\langle A \rangle$ and $\langle AA \rangle$.

For the two-point functions in particular, we found distinctive behaviors at the transition:



[van Egmond et al, SciPost Phys. 12 (2022) 087 & Phys.Rev.D 106 (2022) 7, 074005]

These continuum calculations required some modelling, however, because, the center-symmetric Landau gauge-fixed action is not known in complete form due to the Gribov ambiguity.

These continuum calculations required some modelling, however, because, the center-symmetric Landau gauge-fixed action is not known in complete form due to the Gribov ambiguity.

The results of 2022 used the Curci-Ferrari model to extend phenomenologically the incomplete Faddeev-Popov gauge-fixed action.

In this talk, I would like to convince you of the benefits of the center-symmetric Landau gauge without relying on any type of modelling.

In this talk, I would like to convince you of the benefits of the center-symmetric Landau gauge without relying on any type of modelling.

To this purpose, I shall consider the lattice framework which allows for a full implementation of the center-symmetric Landau gauge.

In this talk, I would like to convince you of the benefits of the center-symmetric Landau gauge without relying on any type of modelling.

To this purpose, I shall consider the lattice framework which allows for a full implementation of the center-symmetric Landau gauge.

I shall consider SU(3) Yang-Mills theory but the analysis can be easily extended to other groups.

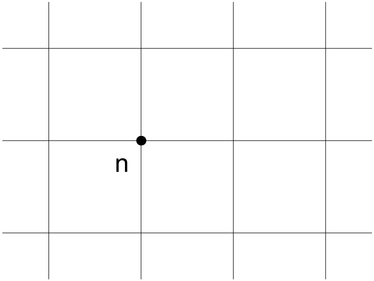
OUTLINE

1. Links, Plaquettes and All That.
2. Polyakov Loop and Center Symmetry.
3. Correlators as Order Parameters?
4. Standard vs Center-Symmetric Landau Gauge.
5. Simulation Results.

1. LINKS, PLAQUETTES, AND ALL THAT

LINK VARIABLES

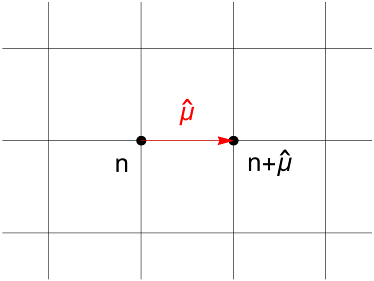
Consider a lattice whose sites are parametrized by a 4-vector n with integer components n_μ, with $\mu = 0, 1, 2, 3$, and such that $0 \leq n_\mu \leq L_\mu$.



LINK VARIABLES

Consider a lattice whose sites are parametrized by a 4-vector n with integer components n_μ, with $\mu = 0, 1, 2, 3$, and such that $0 \leq n_\mu \leq L_\mu$.

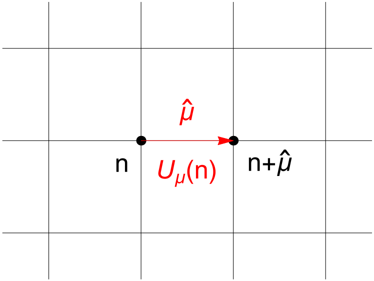
We denote by $\hat{\mu}$ the 4-vector of components $\hat{\mu}_\nu = \delta_{\mu\nu}$. It connects a given site n to its neighbouring site $n + \hat{\mu}$ in the direction μ:



LINK VARIABLES

Consider a lattice whose sites are parametrized by a 4-vector n with integer components n_μ, with $\mu = 0, 1, 2, 3$, and such that $0 \leq n_\mu \leq L_\mu$.

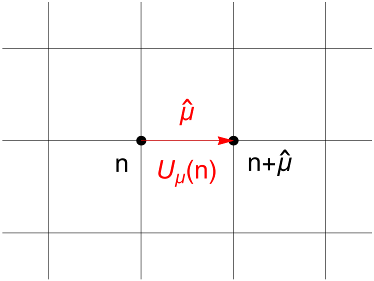
To such an oriented link between neighbouring sites, one associates a link variable $U_\mu(n) \in SU(3)$:



LINK VARIABLES

Consider a lattice whose sites are parametrized by a 4-vector n with integer components n_μ, with $\mu = 0, 1, 2, 3$, and such that $0 \leq n_\mu \leq L_\mu$.

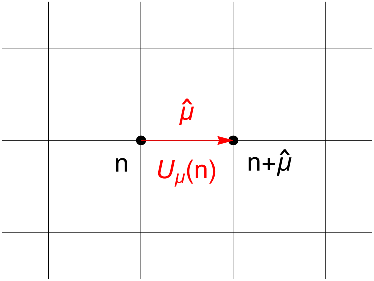
Link variables replace the gauge-field of the continuum set-up:
$U_\mu \sim e^{iaA_\mu} + \mathcal{O}(a^2)$



LINK VARIABLES

Consider a lattice whose sites are parametrized by a 4-vector n with integer components n_μ, with $\mu = 0, 1, 2, 3$, and such that $0 \leq n_\mu \leq L_\mu$.

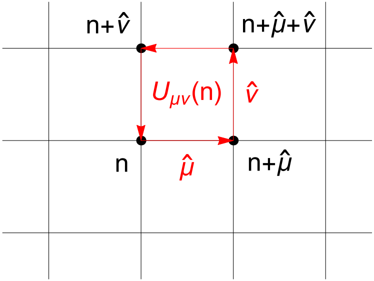
Link variables replace the gauge-field of the continuum set-up:
$U_\mu \sim e^{iaA_\mu} + \mathcal{O}(a^2)$



They are taken periodic in all directions: $U_\mu(n + L_\nu \hat{\nu}) = U_\mu(n)$.

PLAQUETTES AND WILSON ACTION

With the links one can form more general constructs. For instance, to a square loop going through n, $n+\hat{\mu}$, $n+\hat{\mu}+\hat{\nu}$ and $n+\hat{\nu}$:

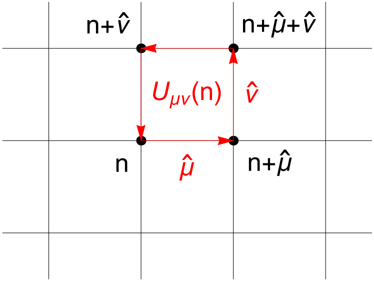


one associates a plaquette variable:

$$U_{\mu\nu}(n) \equiv U_\mu(n)U_\nu(n+\hat{\mu})U^\dagger_\mu(n+\hat{\nu})U^\dagger_\nu(n) \in SU(3)$$

PLAQUETTES AND WILSON ACTION

With the links one can form more general constructs. For instance, to a square loop going through n, $n+\hat{\mu}$, $n+\hat{\mu}+\hat{\nu}$ and $n+\hat{\nu}$:



Plaquettes are the basic variables entering the Wilson action:

$$S_W[U] \equiv \frac{1}{\zeta}\sum_n \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{tr}\left[\mathbb{1} - U_{\mu\nu}(n)\right]$$

a discretized, but still gauge-invariant, version of the Yang-Mills action.

GAUGE INVARIANCE

A gauge transformation is a periodic function $g_0(n)$ on the lattice, taking values in SU(3). It acts on the links as

$$U_\mu(n) \to g_0(n) U_\mu(n) g_0^\dagger(n+\hat{\mu}) \equiv U_\mu^{g_0}(n)$$

GAUGE INVARIANCE

A gauge transformation is a periodic function $g_0(n)$ on the lattice, taking values in SU(3). It acts on the links as

$$U_\mu(n) \to g_0(n) U_\mu(n) g_0^\dagger(n+\hat{\mu}) \equiv U_\mu^{g_0}(n)$$

and on the plaquettes as

$$U_{\mu\nu}(n) \to g_0(n) U_{\mu\nu}(n) g_0^\dagger(n)$$

GAUGE INVARIANCE

A gauge transformation is a periodic function $g_0(n)$ on the lattice, taking values in SU(3). It acts on the links as

$$U_\mu(n) \to g_0(n) U_\mu(n) g_0^\dagger(n+\hat{\mu}) \equiv U_\mu^{g_0}(n)$$

and on the plaquettes as

$$U_{\mu\nu}(n) \to g_0(n) U_{\mu\nu}(n) g_0^\dagger(n)$$

This ensures that the the Wilson action is gauge invariant:

$$S_W[U^{g_0}] = \frac{1}{\zeta} \sum_n \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{tr}\left[\mathbb{1} - g_0(n) U_{\mu\nu}(n) g_0^\dagger(n)\right] = S_W[U]$$

OBSERVABLES

Observables are defined as averages of gauge-invariant functionals, $\mathcal{O}[U^{g_0}] = \mathcal{O}[U]$, with a probability weight given by $e^{-S_W[U]}$:

$$\langle \mathcal{O}[U] \rangle \equiv \frac{\int \mathcal{D}U_\mu \, \mathcal{O}[U] \, e^{-S_W[U]}}{\int \mathcal{D}U_\mu \, e^{-S_W[U]}}$$

OBSERVABLES

Observables are defined as averages of gauge-invariant functionals, $\mathcal{O}[U^{g_0}] = \mathcal{O}[U]$, with a probability weight given by $e^{-S_W[U]}$:

$$\langle \mathcal{O}[U] \rangle \equiv \frac{\int \mathcal{D}U_\mu \, \mathcal{O}[U] \, e^{-S_W[U]}}{\int \mathcal{D}U_\mu \, e^{-S_W[U]}}$$

On the lattice, such integrals are evaluated using Monte-Carlo importance sampling:

– one generates an ensemble of link configurations $\mathcal{E} = \{U_\mu\}$ that follows the probability distribution $e^{-S_W[U]}$;

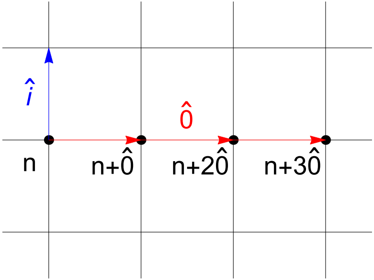
– then, the observables are computed as

$$\langle \mathcal{O}[U] \rangle \simeq \frac{\sum_{U_\mu \in \mathcal{E}} \mathcal{O}[U]}{N_{\rm conf}} \equiv \langle \mathcal{O}[U] \rangle_{\mathcal{E}}$$

2. POLYAKOV LOOP AND CENTER SYMMETRY

POLYAKOV LOOP

Consider a temporal loop on the lattice and let us multiply the corresponding temporal links along the loop:



$$U_0(n)U_0(n+\hat{0})\cdots U_0(n+(L_0-1)\hat{0})$$

POLYAKOV LOOP

Under a gauge transformation $g_0(n)$, this product transforms as:

$$U_0(n)U_0(n+\hat{0})\cdots U_0(n+(L_0-1)\hat{0})$$

$$\downarrow$$

$$g_0(n)U_0(n)\underbrace{g_0^\dagger(n+\hat{0})g_0(n+\hat{0})}_{=\mathbb{1}}U_0(n+\hat{0})\cdots U_0(n+(L_0-1)\hat{0})\underbrace{g_0^\dagger(n+L_0\hat{0})}_{=g_0^\dagger(n)}$$

$$\downarrow$$

$$g_0(n)\,U_0(n)U_0(n+\hat{0})\cdots U_0(n+(L_0-1)\hat{0})\,g_0^\dagger(n)$$

POLYAKOV LOOP

Then, the trace of such a product of temporal links

$$\Phi[U] \equiv \frac{1}{3}\mathrm{tr}\, U_0(n)U_0(n+\hat{0})\cdots U_0(n+(L_0-1)\hat{0})$$

defines a gauge-invariant functional $\Phi[U^{g_0}] = \Phi[U]$, the Polyakov loop, which can be averaged to define an observable $\langle\Phi[U]\rangle$.

POLYAKOV LOOP

Then, the trace of such a product of temporal links

$$\Phi[U] \equiv \frac{1}{3}\mathrm{tr}\, U_0(n)U_0(n+\hat{0})\cdots U_0(n+(L_0-1)\hat{0})$$

defines a gauge-invariant functional $\Phi[U^{g_0}] = \Phi[U]$, the Polyakov loop, which can be averaged to define an observable $\langle\Phi[U]\rangle$.

The physical relevance of $\langle\Phi[U]\rangle$ lies in that

$$-\frac{1}{\beta}\ln\langle\Phi[U]\rangle = \Delta F$$

represents the free-energy of a test color charge in the medium.

POLYAKOV LOOP

Then, the trace of such a product of temporal links

$$\Phi[U] \equiv \frac{1}{3}\mathrm{tr}\, U_0(n)U_0(n+\hat{0})\cdots U_0(n+(L_0-1)\hat{0})$$

defines a gauge-invariant functional $\Phi[U^{g_0}] = \Phi[U]$, the Polyakov loop, which can be averaged to define an observable $\langle\Phi[U]\rangle$.

The physical relevance of $\langle\Phi[U]\rangle$ lies in that

$$-\frac{1}{\beta}\ln\langle\Phi[U]\rangle = \Delta F$$

represents the free-energy of a test color charge in the medium.

$$\text{Thus}: \begin{cases} \text{if } \langle\Phi\rangle = 0, \text{ then } \Delta F = \infty \text{ and the medium confines charges;} \\ \text{if } \langle\Phi\rangle \neq 0, \text{ then } \Delta F < \infty \text{ and the medium accepts charges.} \end{cases}$$

CENTER SYMMETRY

But how could $\langle \Phi[U] \rangle$ vanish in the first place?

CENTER SYMMETRY

But how could $\langle \Phi[U] \rangle$ vanish in the first place?

The Wilson action is symmetric under a more general class of gauge transformations $g(n)$, known as center transformations:

$$g(n + L_0\hat{0}) = e^{\pm i2\pi/3} g(n)$$

For those transformations, one has

$$\Phi[U^g] = e^{\mp i2\pi/3} \Phi[U]$$

CENTER SYMMETRY

But how could $\langle \Phi[U] \rangle$ vanish in the first place?

The Wilson action is symmetric under a more general class of gauge transformations $g(n)$, known as center transformations:

$$g(n + L_0\hat{0}) = e^{\pm i2\pi/3} g(n)$$

For those transformations, one has

$$\Phi[U^g] = e^{\mp i2\pi/3} \Phi[U]$$

This symmetry, when realized, strongly constrains the value of $\langle \Phi[U] \rangle$.

CENTER SYMMETRY CONSTRAINT

Recall that the average is done over the Monte-Carlo ensemble $\mathcal{E}$

$$\langle \Phi[U] \rangle_{\mathcal{E}} = \frac{\sum_{U_\mu \in \mathcal{E}} \Phi[U]}{N_{\rm conf}}$$

CENTER SYMMETRY CONSTRAINT

Recall that the average is done over the Monte-Carlo ensemble $\mathcal{E}$

$$\langle \Phi[U] \rangle_{\mathcal{E}} = \frac{\sum_{U_\mu \in \mathcal{E}} \Phi[U]}{N_{\rm conf}}$$

Let us now introduce a new ensemble $\mathcal{E}^{g^\dagger} = \{U_\mu^{g^\dagger} | U_\mu \in \mathcal{E}\}$.

CENTER SYMMETRY CONSTRAINT

Recall that the average is done over the Monte-Carlo ensemble $\mathcal{E}$

$$\langle \Phi[U] \rangle_{\mathcal{E}} = \frac{\sum_{U_\mu \in \mathcal{E}} \Phi[U]}{N_{\rm conf}}$$

Let us now introduce a new ensemble $\mathcal{E}^{g^\dagger} = \{U_\mu^{g^\dagger} | U_\mu \in \mathcal{E}\}$.
Then

$$\langle \Phi[U] \rangle_{\mathcal{E}} = \frac{\sum_{U_\mu \in \mathcal{E}^{g^\dagger}} \Phi[U^g]}{N_{\rm conf}}$$

CENTER SYMMETRY CONSTRAINT

Recall that the average is done over the Monte-Carlo ensemble $\mathcal{E}$

$$\langle \Phi[U] \rangle_{\mathcal{E}} = \frac{\sum_{U_\mu \in \mathcal{E}} \Phi[U]}{N_{\rm conf}}$$

Let us now introduce a new ensemble $\mathcal{E}^{g^\dagger} = \{U_\mu^{g^\dagger} | U_\mu \in \mathcal{E}\}$. Then

$$\begin{aligned} \langle \Phi[U] \rangle_{\mathcal{E}} &= \frac{\sum_{U_\mu \in \mathcal{E}^{g^\dagger}} \Phi[U^g]}{N_{\rm conf}} \\ &= e^{\mp i 2\pi/3} \frac{\sum_{U_\mu \in \mathcal{E}^{g^\dagger}} \Phi[U]}{N_{\rm conf}} \end{aligned}$$

CENTER SYMMETRY CONSTRAINT

Recall that the average is done over the Monte-Carlo ensemble $\mathcal{E}$

$$\langle \Phi[U] \rangle_{\mathcal{E}} = \frac{\sum_{U_\mu \in \mathcal{E}} \Phi[U]}{N_{\rm conf}}$$

Let us now introduce a new ensemble $\mathcal{E}^{g^\dagger} = \{U_\mu^{g^\dagger} | U_\mu \in \mathcal{E}\}$.
Then

$$\begin{aligned}
\langle \Phi[U] \rangle_{\mathcal{E}} &= \frac{\sum_{U_\mu \in \mathcal{E}^{g^\dagger}} \Phi[U^g]}{N_{\rm conf}} \\
&= e^{\mp i 2\pi/3} \frac{\sum_{U_\mu \in \mathcal{E}^{g^\dagger}} \Phi[U]}{N_{\rm conf}} = e^{\mp i 2\pi/3} \langle \Phi[U] \rangle_{\mathcal{E}^{g^\dagger}}
\end{aligned}$$

What can we deduce from $\langle \Phi[U] \rangle_{\mathcal{E}} = e^{\mp i2\pi/3} \langle \Phi[U] \rangle_{\mathcal{E}^{g^\dagger}}$?

CENTER SYMMETRIC PHASE

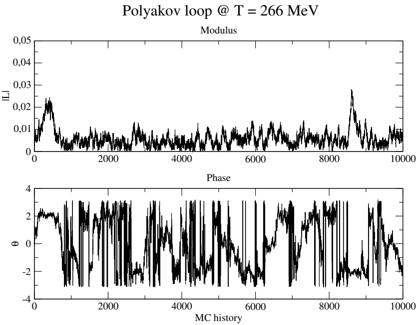
What can we deduce from $\langle \Phi[U] \rangle_{\mathcal{E}} = e^{\mp i2\pi/3} \langle \Phi[U] \rangle_{\mathcal{E}^{g^\dagger}}$?

The MC simulation time is large enough for the system to explore the configuration space symmetrically. In this case, $\mathcal{E}^{g^\dagger} \simeq \mathcal{E}$ and the above identity becomes a constraint on $\langle \Phi[U] \rangle_{\mathcal{E}}$:

$$\langle \Phi[U] \rangle_{\mathcal{E}} \simeq e^{\mp i2\pi/3} \langle \Phi[U] \rangle_{\mathcal{E}}$$

which actually enforces it to be

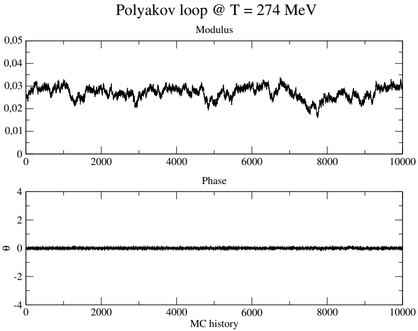
$$\langle \Phi[U] \rangle_{\mathcal{E}} \simeq 0$$



CENTER BROKEN PHASE

What can we deduce from $\langle \Phi[U] \rangle_{\mathcal{E}} = e^{\mp i2\pi/3} \langle \Phi[U] \rangle_{\mathcal{E}^{g^\dagger}}$?

The MC simulation time is not large enough and the system gets essentially stuck in one region of the configuration space such that $\mathcal{E}^{g^\dagger} \neq \mathcal{E}$. In this case, there is no reason for $\langle \Phi[U] \rangle_{\mathcal{E}}$ to vanish and the above identity is just a relation between the averages over distinct regions connected by the symmetry.



TAKE-AWAY MESSAGE

The Polyakov loop $\langle \Phi[U] \rangle_{\mathcal{E}}$ is an order parameter for center symmetry.

Two crucial ingredients:

- $\Phi[U]$ transforms linearly: $\Phi[U^g] = e^{\mp i2\pi/3}\Phi[U]$;
- $\mathcal{E}$ is center-invariant in the symmetric phase: $\mathcal{E}^{g^\dagger} \simeq \mathcal{E}$.

3. CORRELATORS AS ORDER PARAMETERS?

LINK CORRELATORS

Suppose now that we want to compute link correlators:

$$\langle U_{\mu_1}(n_1)\otimes\cdots\otimes U_{\mu_p}(n_p)\rangle$$

from which we could eventually access gauge-field correlators.

LINK CORRELATORS

Suppose now that we want to compute link correlators:

$$\langle U_{\mu_1}(n_1) \otimes \cdots \otimes U_{\mu_p}(n_p) \rangle$$

from which we could eventually access gauge-field correlators.

Could any of those be used as an order parameters for center symmetry?

LINK CORRELATORS

Suppose now that we want to compute link correlators:

$$\langle U_{\mu_1}(n_1) \otimes \cdots \otimes U_{\mu_p}(n_p) \rangle$$

from which we could eventually access gauge-field correlators.

Could any of those be used as an order parameters for center symmetry?

I shall focus on the one-point function $\langle U_\mu(n) \rangle$ for simplicity and slightly touch on the two-point function $\langle U_{\mu_1}(n_1) \otimes U_{\mu_2}(n_2) \rangle$.

GAUGE FIXING

Important warning: the average $\langle U_\mu(n) \rangle$ is not defined over the original ensemble $\mathcal{E}$ but rather over a gauge-fixed ensemble $\mathcal{E}_F$:

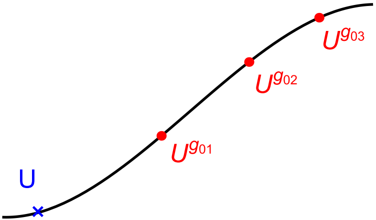
$$\langle U_\mu(n) \rangle_{\mathcal{E}_F} \equiv \frac{\sum_{U_\mu \in \mathcal{E}_F} U_\mu(n)}{N_{\rm conf}}$$

GAUGE FIXING

Important warning: the average $\langle U_\mu(n)\rangle$ is not defined over the original ensemble $\mathcal{E}$ but rather over a gauge-fixed ensemble $\mathcal{E}_F$:

$$\langle U_\mu(n)\rangle_{\mathcal{E}_F} \equiv \frac{\sum_{U_\mu\in\mathcal{E}_F} U_\mu(n)}{N_{\rm conf}}$$

In practice, $\mathcal{E}_F$ is obtained by maximizing a given functional $F[U]$ along the gauge-orbits of the configurations in $\mathcal{E}$.



Starting from $U_\mu\in\mathcal{E}$, one explores the gauge orbit $U_\mu^{g_0}$, until a g_0 is found that maximizes (locally) $F[U^{g_0}]$.

SYMMETRY CONSTRAINT

Let us now proceed as in the case of the Polyakov loop and write

$$\begin{aligned}
\langle U_\mu(n)\rangle_{\mathcal{E}_F} &= \frac{\sum_{U_\mu\in\mathcal{E}_F} U_\mu(n)}{N_{\rm conf}}\\
&= \frac{\sum_{U_\mu\in\mathcal{E}_F^{g^\dagger}} U^g_\mu(n)}{N_{\rm conf}}\\
&= g(n)\frac{\sum_{U_\mu\in\mathcal{E}_F^{g^\dagger}} U_\mu(n)}{N_{\rm conf}}g^\dagger(n+\hat\mu)\\
&= g(n)\langle U_\mu(n)\rangle_{\mathcal{E}_F^{g^\dagger}}\, g^\dagger(n+\hat\mu)\,.
\end{aligned}$$

SYMMETRY CONSTRAINT

The identity

$$\langle U_\mu(n)\rangle_{\mathcal{E}_F} = g(n)\langle U_\mu(n)\rangle_{\mathcal{E}_F^{g^\dagger}} g^\dagger(n+\hat{\mu})$$

looks pretty similar to

$$\langle \Phi[U]\rangle_{\mathcal{E}} = e^{\mp i2\pi/3}\langle \Phi[U]\rangle_{\mathcal{E}^{g^\dagger}}$$

It is a linear relation connecting averages over two ensembles.

SYMMETRY CONSTRAINT

The identity

$$\langle U_\mu(n)\rangle_{\mathcal{E}_F} = g(n)\langle U_\mu(n)\rangle_{\mathcal{E}_F^{g^\dagger}} g^\dagger(n+\hat{\mu})$$

looks pretty similar to

$$\langle \Phi[U]\rangle_{\mathcal{E}} = e^{\mp i2\pi/3}\langle \Phi[U]\rangle_{\mathcal{E}^{g^\dagger}}$$

It is a linear relation connecting averages over two ensembles.

Can we deduce that $\langle U_\mu(n)\rangle_{\mathcal{E}_F}$ is an order parameter for center symmetry?

SYMMETRY CONSTRAINT

The identity

$$\langle U_\mu(n)\rangle_{\mathcal{E}_F} = g(n)\langle U_\mu(n)\rangle_{\mathcal{E}_F^{g^\dagger}} g^\dagger(n+\hat{\mu})$$

looks pretty similar to

$$\langle \Phi[U]\rangle_{\mathcal{E}} = e^{\mp i2\pi/3}\langle \Phi[U]\rangle_{\mathcal{E}^{g^\dagger}}$$

It is a linear relation connecting averages over two ensembles.

Can we deduce that $\langle U_\mu(n)\rangle_{\mathcal{E}_F}$ is an order parameter for center symmetry?

The answer actually depends on F!

GENERIC GAUGE

A crucial ingregient in the case of $\langle \Phi[U] \rangle_{\mathcal{E}}$ was that $\mathcal{E}^{g^\dagger} \simeq \mathcal{E}$ in the symmetric phase.

GENERIC GAUGE

A crucial ingregient in the case of $\langle \Phi[U] \rangle_{\mathcal{E}}$ was that $\mathcal{E}^{g^\dagger} \simeq \mathcal{E}$ in the symmetric phase.

In the case of $\langle U_\mu \rangle_{\mathcal{E}_F}$, do we have $\mathcal{E}_F^{g^\dagger} \simeq \mathcal{E}_F$?

GENERIC GAUGE

A crucial ingregient in the case of $\langle \Phi[U] \rangle_{\mathcal{E}}$ was that $\mathcal{E}^{g^\dagger} \simeq \mathcal{E}$ in the symmetric phase.

In the case of $\langle U_\mu \rangle_{\mathcal{E}_F}$, do we have $\mathcal{E}_F^{g^\dagger} \simeq \mathcal{E}_F$?

A priori not: the configurations of $\mathcal{E}_F^{g^\dagger}$ maximize the functional $F[U^g]$ which, for a generic choice of $F[U]$, has no reason to be equal to $F[U]$.

GENERIC GAUGE

Then, for a generic choice of gauge, the configurations of $\mathcal{E}_F$ and $\mathcal{E}_F^{g^\dagger}$ correspond to different gauges, and the identity

$$\langle U_\mu(n)\rangle_{\mathcal{E}_F} = g(n)\langle U_\mu(n)\rangle_{\mathcal{E}_F^{g^\dagger}} g^\dagger(n+\hat{\mu})$$

just relates the one-point functions in two different gauges.

GENERIC GAUGE

Then, for a generic choice of gauge, the configurations of $\mathcal{E}_F$ and $\mathcal{E}_F^{g^\dagger}$ correspond to different gauges, and the identity

$$\langle U_\mu(n)\rangle_{\mathcal{E}_F} = g(n)\langle U_\mu(n)\rangle_{\mathcal{E}_F^{g^\dagger}} g^\dagger(n+\hat{\mu})$$

just relates the one-point functions in two different gauges.

In this generic case, there is no actual constraint on $\langle U_\mu(n)\rangle_{\mathcal{E}_F}$ which is then no order parameter for center symmetry.

CENTER-SYMMETRIC GAUGE

But what happens if a gauge functional $F[U]$ is found such that

$$F[U^g] = F[U]?$$

CENTER-SYMMETRIC GAUGE

In this case, the configurations of $\mathcal{E}_F^{g^\dagger}$ and $\mathcal{E}_F$ agree since they maximize the same functional $F[U^g] = F[U]$.

CENTER-SYMMETRIC GAUGE

In this case, the configurations of $\mathcal{E}_F^{g^\dagger}$ and $\mathcal{E}_F$ agree since they maximize the same functional $F[U^g] = F[U]$.

The previous identity becomes

$$\langle U_\mu(n)\rangle_{\mathcal{E}_F} = g(n)\,\langle U_\mu(n)\rangle_{\mathcal{E}_F}\, g^\dagger(n+\hat{\mu})$$

that is a constraint on the possible values of $\langle U_\mu(n)\rangle_{\mathcal{E}_F}$.

CENTER-SYMMETRIC GAUGE

In this case, the configurations of $\mathcal{E}_F^{g^\dagger}$ and $\mathcal{E}_F$ agree since they maximize the same functional $F[U^g] = F[U]$.

The previous identity becomes

$$\langle U_\mu(n)\rangle_{\mathcal{E}_F} = g(n)\,\langle U_\mu(n)\rangle_{\mathcal{E}_F}\, g^\dagger(n+\hat{\mu})$$

that is a constraint on the possible values of $\langle U_\mu(n)\rangle_{\mathcal{E}_F}$.

In this particular gauge, $\langle U_\mu(n)\rangle_{\mathcal{E}_F}$ becomes a potential order parameter for center-symmetry.

4. STANDARD VS CENTER-SYMMETRIC LANDAU GAUGE

STANDARD LANDAU (SL) GAUGE

In the standard Landau (SL) gauge, the gauge-fixed configurations are obtained by maximizing the functional

$$F_{\mathrm{SL}}[U] \equiv \mathrm{Re}\,\mathrm{tr}\sum_{n,\mu} U_\mu(n)$$

This functional is not invariant under center transformations

$$\begin{aligned} F_{\mathrm{SL}}[U^g] &= \mathrm{Re}\,\mathrm{tr}\sum_{n,\mu} g(n)U_\mu(n)g^\dagger(n+\hat{\mu}) \\ &= \mathrm{Re}\,\mathrm{tr}\sum_{n,\mu} g^\dagger(n+\hat{\mu})g(n)U_\mu(n) \neq F_{\mathrm{SL}}[U] \end{aligned}$$

Therefore, there is no reason for $\langle U_\mu(n)\rangle_{\mathcal{E}_{\mathrm{SL}}}$ to qualify as an order parameter for center symmetry.

STANDARD LANDAU (SL) GAUGE

In fact the functional $F_{\rm SL}[U]$ has already a number of symmetries:

- color rotations: $U_\mu(n) \to e^{i\theta^a t^a} U_\mu(n) e^{-i\theta^a t^a}$;
- charge conjugation: $U_\mu(n) \to U^*_\mu(n)$.

that impose $\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm SL}} = \eta\mathbb{1}$, with $\eta \in \mathbb{R}$ and independent of n from translation invariance. This implies

$$\frac{\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm SL}}}{(\det \langle U_\mu(n)\rangle_{\mathcal{E}_{\rm SL}})^{1/3}} = \mathbb{1}$$

irrespectively of the fate of center symmetry!

CENTER-SYMMETRIC LANDAU (CSL) GAUGE

Consider now this other functional

$$F_{\rm CSL}[U] \equiv {\rm Re}\,{\rm tr}\sum_\mu g_c^\dagger(\hat{\mu})\,U_\mu(n)\,, \text{ with } g_c(\hat{\mu}) = e^{-i\frac{4\pi}{3}\frac{\hat{\mu}_0}{L_0}\frac{\lambda^3}{2}}$$

We have a remnant of the previous symmetries:

- Abelian color rotations: $U_\mu(n) \to e^{i\theta^j t^j} U_\mu(n) e^{-i\theta^j t^j}$, with $t^j \in \{\lambda_3/2, \lambda_8/2\}$ the Abelian generators;
- charge conjugation: $U_\mu(n) \to e^{i\pi\lambda_1/2} U_\mu^*(n) e^{-i\pi\lambda_1/2}$.

CENTER-SYMMETRIC LANDAU (CSL) GAUGE

But more importantly, $F_{\rm CSL}[U]$ is invariant under the center transformation

$$g(n)=e^{i\pi\frac{\lambda_4}{2}}e^{i\pi\frac{\lambda_1}{2}}e^{-i\frac{n_0}{L_0}\pi\left(\lambda_3+\frac{\lambda_8}{\sqrt{3}}\right)}$$

Let us check that this is indeed a center transformation:

$$\begin{aligned}
g(n+L_0\hat{0}) &= e^{i\pi\frac{\lambda_4}{2}}e^{i\pi\frac{\lambda_1}{2}}e^{-i\frac{n_0}{L_0}\pi\left(\lambda_3+\frac{\lambda_8}{\sqrt{3}}\right)}e^{-i\pi\left(\lambda_3+\frac{\lambda_8}{\sqrt{3}}\right)}\\
&= g(n)\exp\left\{-i\pi\begin{pmatrix} 1+\frac{1}{3} & 0 & 0\\ 0 & -1+\frac{1}{3} & 0\\ 0 & 0 & -\frac{2}{3}\end{pmatrix}\right\}\\
&= e^{i\frac{2\pi}{3}}g(n)\,.
\end{aligned}$$

CENTER-SYMMETRIC LANDAU (CSL) GAUGE

Let us now check that $g(n)$ leaves the functional $F_{\rm CSL}[U]$ invariant:

$$\begin{aligned} F_{\rm CSL}[U^g] &= \sum_{n,\mu} {\rm Re}\,{\rm tr}\, g_c^\dagger(\hat{\mu}) g(n) U_\mu(n) g^\dagger(n+\hat{\mu}) \\ &= \sum_{n,\mu} {\rm Re}\,{\rm tr}\, g^\dagger(n+\hat{\mu}) g_c^\dagger(\hat{\mu}) g(n) U_\mu(n) \end{aligned}$$

It is easily verified that

$$g^\dagger(n+\hat{\mu}) g_c^\dagger(\hat{\mu}) g(n) = g_c^\dagger(\hat{\mu})$$

and thus

$$F_{\rm CSL}[U^g] = F_{\rm CSL}[U]$$

and $\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}}$ becomes a potential order parameter for center sym.

ORDER PARAMETER

In fact $\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}}$ obeys the constraint

$$\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}} = g(n)\,\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}}\,g^\dagger(n+\hat{\mu})$$

which is solved as

$$\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}} = \eta\, g_c(\hat{\mu}) = \eta\, e^{-i\frac{4\pi}{3}\frac{\hat{\mu}_0}{L_0}\frac{\lambda^3}{2}}$$

with η real, and independent of n from translation invariance.

ORDER PARAMETER

In fact $\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}}$ obeys the constraint

$$\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}} = g(n)\,\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}}\,g^\dagger(n+\hat{\mu})$$

which is solved as

$$\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}} = \eta\,g_c(\hat{\mu}) = \eta\,e^{-i\frac{4\pi}{3}\frac{\hat{\mu}_0}{L_0}\frac{\lambda^3}{2}}$$

with η real, and independent of n from translation invariance.

This implies

$$\frac{\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}}}{(\det\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}})^{1/3}} = e^{-i\frac{4\pi}{3}\frac{\hat{\mu}_0}{L_0}\frac{\lambda^3}{2}}$$

ORDER PARAMETER

In fact $\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}}$ obeys the constraint

$$\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}} = g(n)\,\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}}\,g^\dagger(n+\hat{\mu})$$

which is solved as

$$\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}} = \eta\, g_c(\hat{\mu}) = \eta\, e^{-i\frac{4\pi}{3}\frac{\hat{\mu}_0}{L_0}\frac{\lambda^3}{2}}$$

with η real, and independent of n from translation invariance.

This implies

$$\frac{\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}}}{(\det\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm CSL}})^{1/3}} = e^{-i\frac{4\pi}{3}\frac{\hat{\mu}_0}{L_0}\frac{\lambda^3}{2}} \quad \left[\text{vs}\ \frac{\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm SL}}}{(\det\langle U_\mu(n)\rangle_{\mathcal{E}_{\rm SL}})^{1/3}} = \mathbb{1}\right]$$

5. SIMULATION RESULTS

CSL LINK AVERAGE

In the CSL gauge, the claim is that (L_0 is the temporal extent)

$$M \equiv \frac{\langle U_0(n)\rangle_{\mathcal{E}_{\rm CSL}}}{(\det\langle U_0(n)\rangle_{\mathcal{E}_{\rm CSL}})^{1/3}} = \begin{pmatrix} e^{-i\frac{2\pi}{3L_0}} & & \\ & e^{i\frac{2\pi}{3L_0}} & \\ & & 1 \end{pmatrix}$$

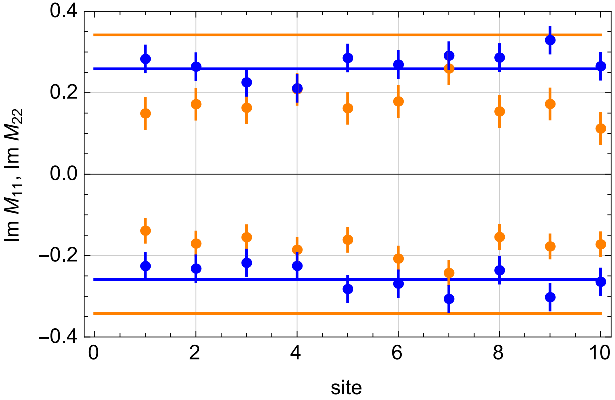
is an order parameter for center symmetry.

To test this, we generated two ensembles of 100 CSL configurations:

- one above the transition, corresponding to $L_0 = 6$,
- one below the transition, corresponding to $L_0 = 8$.

CSL LINK AVERAGE

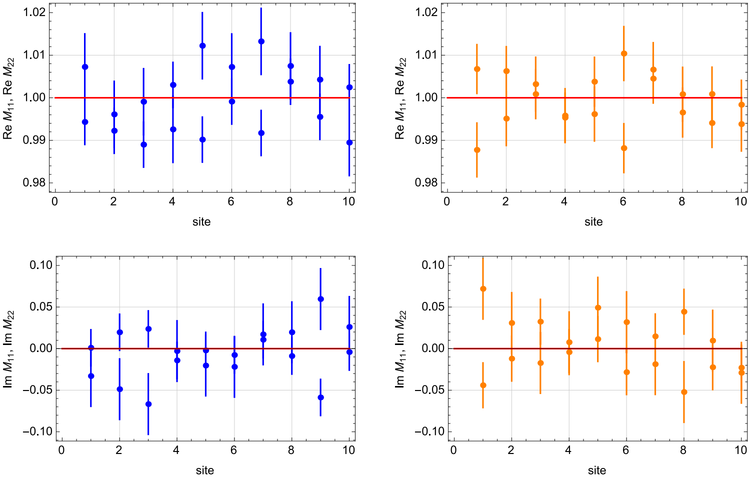
We looked at the imaginary parts of M_{11} and M_{22} and compared to the symmetry constraints, over various lattice sites (10 here):



Im M_{11} and Im M_{22} behave as order parameters!

SL LINK AVERAGE

Landau gauge data compatible with $M \equiv \langle U_0 \rangle / (\det \langle U_0 \rangle)^{1/3} = \mathbb{1}$:



Not an order parameter in this case!

A WORD ON THE TWO-LINK CORRELATORS

Similar symmetry constraints can be derived for higher correlators.

It is conventient to work with twisted link variables:

$$\tilde{U}_\mu(n) \equiv U_\mu^{g_c}(n) \quad \text{with} \quad g_c(n) = e^{-i\frac{4\pi}{3}\frac{n_0}{L_0}\frac{\lambda^3}{2}}$$

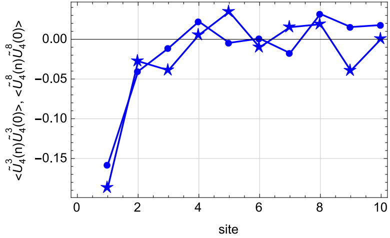
One finds for instance that, in the center-symmetric phase

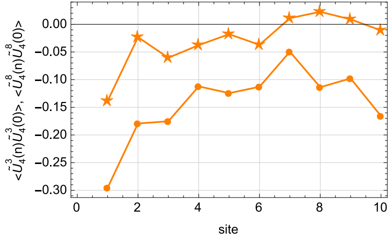
$$\langle(\mathrm{tr}\lambda_3\tilde{U}_0(n_1))(\mathrm{tr}\lambda_3\tilde{U}_0(n_2))\rangle_{\rm CSL} = \langle(\mathrm{tr}\lambda_8\tilde{U}_0(n_1))(\mathrm{tr}\lambda_8\tilde{U}_0(n_2))\rangle_{\rm CSL}$$

No other symmmetry ensures this when center symmetry is broken (as opposed to the SL gauge).

A WORD ON THE TWO-LINK CORRELATORS

Preliminary results:





See Paulo Silva's talk for the connection to the gluon propagator in the center-symmetric Landau gauge.

INVARIANCE OF THE GAUGE-FIXED ENSEMBLE

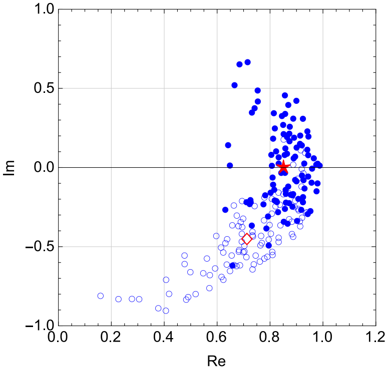
The above discussion/results relied on $\mathcal{E}_F^{g^\dagger} \simeq \mathcal{E}_F$ in the symmetric phase.

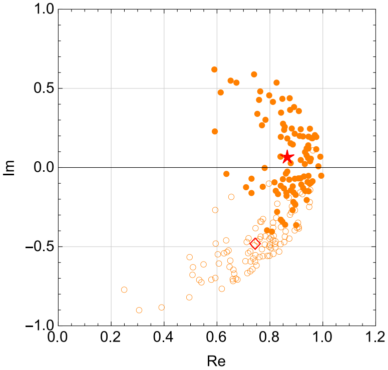
This might be difficult to realize, even in the CSL gauge because $\mathcal{E}_F$ is actually not the ensemble of ALL local maxima, but just a bunch of them selected on each gauge orbit.

Let us test how good is $\mathcal{E}_F^{g^\dagger} \simeq \mathcal{E}_F$ satisfied (for our modest 100 configuration ensembles).

NON-INVARIANCE OF THE SL GAUGE ENSEMBLE

Gauge-fixed ensemble $\mathcal{E}_{\rm SL}$ (plain dots) vs its center-transformed version $\mathcal{E}_{\rm SL}^{g^\dagger}$ (empty dots):

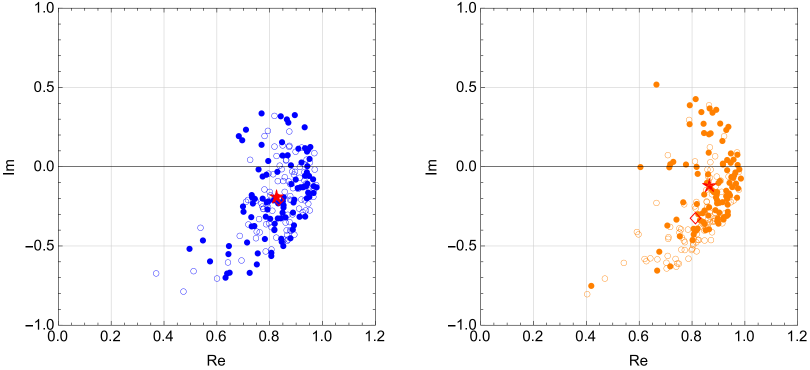




The gauge-fixed ensemble is not center-symmetric, neither at low T nor at high T.

INVARIANCE OF THE CSL GAUGE ENSEMBLE

Gauge-fixed ensemble $\mathcal{E}_{\mathrm{CSL}}$ (plain dots) vs its center-transformed version $\mathcal{E}_{\mathrm{CSL}}^{g^\dagger}$ (empty dots):



The gauge-fixed ensemble looks approximately center-symmetric at low T.

CONCLUSIONS

- The question of knowing whether gauge-field or gauge-link correlators can be used as order parameter for center symmetry depends on the gauge.

- SL gauge correlators do not qualify as order parameters because center symmetry does not constrain their values, but rather relates them to correlators in another gauge.

- In contrast, correlators defined in the CSL gauge are constrained by the symmetry and, therefore, qualify as order parameters.

- This places the CSL gauge as a privileged framework for functional calculations at finite temperature.

- We have illustrated these considerations by implementing the CSL gauge on the lattice and by evaluating the one-link average as well as the two-link correlator which both show order parameter patterns.

OUTLOOK

- Gluon propagator in the CSL gauge, see next talk.
- Simulate other temperatures.
- Increase the statistics, in particular to test $\mathcal{E}_{\rm CSL}^{g^\dagger} \simeq \mathcal{E}_{\rm CSL}$.
- SU(2), include quarks, higher order correlators.

THANK YOU!